ON THE LOCAL STRUCTURE THEOREM AND EQUIVARIANT GEOMETRY OF COTANGENT VECTOR BUNDLES

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ABSTRACT. Let G be a connected reductive group acting on an irreducible normal algebraic variety X. We give a slightly improved version of local structure theorems obtained by F.Knop and D.A.Timashev that describe an action of some parabolic subgroup of G on an open subset of X. We also extend various results of E.B.Vinberg and D.A.Timashev on the set of horospheres in X. We construct a family of nongeneric horospheres in X and a variety $\mathcal{H}or$ parameterizing this family, such that there is a rational G-equivariant symplectic covering of cotangent vector bundles $T^*\mathcal{H}or \dashrightarrow T^*X$. As an application we get a description of the image of the moment map of T^*X obtained by F.Knop by means of geometric methods that do not involve differential operators.

Let G be a connected reductive group acting on an irreducible normal algebraic variety X. In this paper we discuss various results describing the action of a certain parabolic subgroup of G on an open subset of X. These results are usually called "local structure theorems". The first results of that kind were discovered by F.Grosshans [5], and independently by M.Brion, D.Luna and T.Vust [3]. We should mention that the latter theorem was improved by F.Knop [10]. He used it to integrate the invariant collective motion and to describe the closures of so-called generic flats for the class of varieties that he called non-degenerate. (We recall the definition later.) In [17] D.A.Timashev proves a generalization of the local structure theorem and this allows him to integrate the invariant collective motion (generalizing the ideas of F.Knop [10] with a weaker assumption than non-degeneracy). In this paper we give an refined version of the local structure theorem obtained by D.A.Timashev. One of the applications of this theorem is to study the closures of generic flats for some class of varieties, this will be published elsewhere.

The second aim of this paper is to generalize a result of E.B.Vinberg [18] who constructed a rational Galois cover of T^*X for a quasiaffine X by the cotangent bundle to the variety of generic horospheres (this results are also valid for non-degenerate varieties). By horospheres we call the orbits of all maximal unipotent subgroups of G in X. It can be observed that the set of generic horospheres (i.e. the generic orbits of maximal unipotent subgroups of G) can be supplied a structure of algebraic variety. The Galois group of this rational cover is equal to the little Weyl group of the variety X. It is well known that this result could not be directly generalized to arbitrary varieties since the set of generic horospheres is not good enough for this purpose, as can be seen in the case when X is a flag variety. These results were substantially generalized by D.A.Timashev for some class of varieties which is wider than non-degenerate varieties and which included flag varieties (however the former class does not contain all horospherical varieties). In this paper we construct a family of degenerate horospheres and a variety $\mathcal{H}or$ parameterizing them, such

that there is a rational covering of the cotangent vector bundles $T^*\mathcal{H}or \dashrightarrow T^*X$. It is proved that the Galois group of this rational covering is the little Weyl group introduced by F.Knop [9].

The structure of this paper is the following. Section 1 is preliminary, we recall the local structure theorem introduced by F.Knop and its corollaries. In Section 2 we construct a Q-equivariant mapping π_D from an open subset $\mathring{X} \subset X$ to a generalized flag variety of a Levi subgroup of Q (here Q is the common stabilizer of the divisors of B-semi-invariant rational functions, which is a parabolic subgroup of G). In Section 3 we relate the fibers of the introduced mapping π_D to a cross section introduced by F.Knop. The map π_D allows us to give a refined version of the local structure theorem in the sense of D.A.Timashev in Section 4. In Section 5 using ideas of F.Knop [12] of studing Białynicki-Birula cells we construct a foliation of nongeneric horospheres such that the G-translate of the conormal bundle to this foliation is dense in T^*X. We note that in the situations closely related to the topic of the present paper the Białynicki-Birula decomposition was also used by D.Luna [14] and M.Brion [2]. Section 6 is devoted to generalization of the construction of E.B.Vinberg that relates T^*X and the cotangent bundle to the constructed foliation of horospheres. In Section 7 we prove that the Galois group of the rational covering $T^*\mathcal{H}or \dashrightarrow T^*X$ is equal to the little Weyl group W_X. We also give an elementary description of the image of the normalized moment map, we note that our proof does not involve differential operators (cf. Knop [9],[12]). This work should be considered as a direct continuation of [10],[18],[17].

The author is grateful to D.A.Timashev for fruitful discussions that lead to simplification of most of the proofs [1] and for explaining the formula for the normalized moment map. I express my gratitude to M.Brion for useful discussions and for careful reading of this work. I am grateful to F.Knop for his idea of using Białynicki-Birula cells in Section 5. I would like to thank the referees for their useful comments that considerably improved the exposition of the paper.

Notation and conventions.

All varieties are considered over an algebraically closed field $\mathbb{K}$ of characteristic zero. By Gothic letters we denote Lie algebras corresponding to algebraic groups, denoted by similar capital Latin letters. Let us choose a G-invariant nondegenerate quadratic Cartan-Killing form on the algebra $\mathfrak{g}$ as a trace form induced from a faithful representation of G. This form identifies $\mathfrak{g}$ and $\mathfrak{g}^*$. Speaking of the action $G : \mathfrak{g}$ (resp. $G : \mathfrak{g}^*$) we always assume that it is (co)adjoint. For $\mathfrak{h} \subset \mathfrak{g}$ by $\mathfrak{h}^\perp$ we denote the annihilator of $\mathfrak{h}$ in $\mathfrak{g}^*$.

We fix a Borel subgroup $B \subset G$, and a maximal torus $T \subset B$. Let B^- be the unique Borel subgroup of G such that $B^- \cap B = T$. By $P \supset B$ we denote a parabolic subgroup. By $P^- \supset B^-$ we denote the parabolic subgroup opposite to P. Denote by P_u (resp. P_u^-) the unipotent radical of P (resp. P^-).

Let $\Xi = \Xi(T)$ be the lattice of characters of T. Consider the lattice of one parameter subgroups $\Lambda = \Lambda(T)$. For $\lambda : \mathbb{K}^* \to T$ and a character $\chi \in \Xi$ we have a pairing $\langle \lambda, \chi \rangle$ defined by the formula $\chi(\lambda(t)) = t^{\langle \lambda, \chi \rangle}$ that identifies Λ and Ξ^*. We shall denote the element in Ξ^* corresponding to λ by the same letter, this should not lead to confusion. We use an additive notation for the group law in Λ and in Ξ.

Denote by $W = N_G(T)/T$ the Weyl group of G. Δ is the root system of the Lie algebra $\mathfrak{g}$ corresponding to T. $\Delta^+(\Delta^-)$ is the system of positive (negative)

roots corresponding to the Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$. Π is the system of simple roots. We also have the standard decomposition $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ into the root subspaces.

For $\alpha \in \Delta$ let $e_\alpha \in \mathfrak{g}_\alpha$ be the corresponding element of a Chevalley basis, $\alpha^\vee$ be the corresponding coroot, and s_α be the corresponding reflection. $w_0 \in W$ is the longest element in the Weyl group. By $\mathfrak{t}^*/W$ we denote the corresponding geometric quotient of $\mathfrak{t}^*$ by W. Let L be a Levi subgroup of a parabolic subgroup $P \supset B$. Assume that L contains T. Then $B_L = L \cap B$ is a Borel subgroup of L. We denote by $\Delta_L \subset \Delta$ and by $\Delta_L^+ \subset \Delta_L$ the root subsystem corresponding to L and its set of positive roots. The subset of simple roots in Δ_L^+ is denoted by Π_L. We denote by $C_L \subset \Xi(T)$ the dominant Weyl chamber of L with respect to the positive root system Δ_L^+. By C_L° we denote the interior of C_L. For a parabolic subgroup P (resp. $\mathfrak{p}$) containing T (resp.$\mathfrak{t}$) we denote by $\Delta(P_u)$ (resp. $\Delta(\mathfrak{p}_u)$) the subset of roots in Δ corresponding to the root decomposition of $\mathfrak{p}_u$.

Consider the simple G-module V_χ with highest weight χ and its dual V_χ^*. The highest weight vector of V_χ is denoted by σ_χ and the lowest weight vector of V_χ^* is denoted by $\sigma_{-\chi}^*$, $\langle v, w \rangle$ is the pairing of $v \in V_\chi$ and $w \in V_\chi^*$. By $\mathrm{Wt}(V_\chi)$ we denote the set of weights for the T-action on V_χ.

For an algebraic group H by the superscript $(-)^{(H)}$ we mean H-semi-invariants and by $(-)_\chi^{(H)}$ we mean H-semi-invariants of weight χ.

Let $G \supset H$ be linear algebraic groups and Z be a quasiprojective H-variety. Assume that Z is normal or the quotient map $G \to G/H$ is locally trivial for Zariski topology. Then we may form a quasi-projective G-variety $G *_H Z$, by considering the quotient of $G \times Z$ by the action of H: $(g, z) \mapsto (gh^{-1}, hz)$. The image of a point (g, z) in this quotient we shall denote by $[g * z]$.

For an algebraic group action G on X, ξx is the velocity vector of $\xi \in \mathfrak{g}$ at $x \in X$, $\mathfrak{g}x$ is the tangent space to the orbit Gx in x and G_x is the stabilizer of x. For affine X and a group G, in the case when the algebra $\mathbb{K}[X]^G$ of G-invariant regular functions on X is finitely generated, by $X /\!\!/ G$ we denote a quotient of X which is equal to $\mathrm{Spec}\, \mathbb{K}[X]^G$. If the variety X is smooth we can define the moment map $\mu_X : T^*X \longrightarrow \mathfrak{g}^*$ (where T^*X is the cotangent bundle of X) by the following formula.

$$\langle \mu_X(\alpha), \xi \rangle = \langle \alpha, \xi x \rangle, \ \forall x \in X, \ \alpha \in T_x^*X, \ \xi \in \mathfrak{g}.$$

Let us recall that for a homogenous variety $X = G/H$ the cotangent bundle can be expressed as:

$$T^*X \cong G *_H (\mathfrak{g}/\mathfrak{h})^* \cong G *_H \mathfrak{h}^\perp.$$

The moment map is induced by the inclusion $\mathfrak{h}^\perp \hookrightarrow \mathfrak{g}^*$. Its image is equal to $G\mathfrak{h}^\perp$.

1. Local structure theorem

We begin with some preliminary remarks. Consider a normal G-variety X and a Cartier divisor $D = \sum a_i D_i$, where D_i are B-stable prime Cartier divisors. Let us call D a B-divisor. We denote by $P[D_i]$ the stabilizer of D_i. The stabilizer of the B-divisor D is defined as the intersection of the stabilizers of its prime components $P[D] = \bigcap_i P[D_i]$ (it is clearly a parabolic subgroup of G). As we have $P[D + D'] = P[D] \cap P[D']$, there exists a B-divisor for which $P[D]$ is absolutely minimal. We denote this parabolic subgroup by $P(X)$.

Replacing a divisor D with a sufficiently large multiple nD, we may assume that D is G-linearized ([8]), and in particular B-linearized. Any two G-linearizations differ by a character of G, we choose one of them.

To every B-divisor $D = \sum a_i D_i$ we can associate a weight in the following way. If D is effective and G-linearized we have a section $\sigma_D \in H^0(X, \mathcal{O}(D))_\chi^{(B)}$ whose scheme of zeros is D, then σ_D is a B-semi-invariant vector of some weight χ. Choose

integers n_i such that the divisors $n_i D_i$ are G-linearized and the corresponding sections σ_{D_i} have weights χ_{D_i}. To a B-divisor $D = \sum a_i D_i$ we associate the rational weight $\chi_D = \sum \frac{a_i}{n_i} \chi_{D_i}$.

Definition 1.1. [10] Consider a B-divisor $D = \sum a_i D_i$. Let χ_D be the weight of this divisor. We call χ_D $P[D]$-regular if $\langle \chi_D, \alpha^\vee \rangle \neq 0$ for all $\alpha \in \Delta(P[D]_u)$.

Remark 1.2. We recall that $\langle \chi_D, \alpha^\vee \rangle = 0$ for all $\alpha \in \Delta_{L[D]}$, where $L[D] \subset P[D]$ is the Levi subgroup containing T. We also note that every effective divisor D with stabilizer $P[D]$ is $P[D]$-regular. This is a standard fact of the representation theory applied to the B-semi-invariant section $\sigma_D \in H^0(X, \mathcal{O}(D))_{\chi_D}^{(B)}$.

Let us recall from [10] the definition of non-degenerate varieties:

Definition 1.3. A G-variety X is called non-degenerate if there exists a rational B-semi-invariant function $f_\chi \in \mathbb{K}(X)_\chi^{(B)}$ with divisor $D = (f_\chi)$ such that χ is $P(X)$-regular.

Given a B-divisor, we can consider the following $P[D]$-equivariant map ([10]):

$$\psi_D : X \setminus D \longrightarrow \mathfrak{g}^* : \ x \mapsto l_x, \ \ where \ \ l_x(\xi) = \sum_i a_i \frac{\xi \sigma_i}{\sigma_i}(x).$$

By the next lemma we may assume that the B-invariant divisor in consideration is Cartier.

Lemma 1.4. ([10, Lemma 2.2]) Let X be a normal G-variety and $D \subset X$ a prime divisor. Then D is a Cartier divisor outside $Y = \bigcap_{g \in G} gD$.

Later for D we shall take a B-invariant and not G-invariant divisor. Thus Y is a proper subset of D, so we may shrink the variety and consider $X \setminus Y$ instead of X.

Let us recall the version of the local structure theorem obtained by Knop.

Theorem 1.5. ([10, Thm. 2.3, Prop. 2.4]) Let X be a normal G-variety with a B-divisor D. Assume that χ_D is $P[D]$-regular. Then:
(i) The image of ψ_D is a single $P[D]$-orbit equal to $\chi_D + \mathfrak{p}_u$.
(ii) For some $x_0 \in X \setminus D$ let

$$\eta_0 := \psi_D(x_0), \quad L := G_{\eta_0}, \quad Z_0 := \psi_D^{-1}(\eta_0).$$

Then L is a Levi subgroup of $P[D]$ and there is an isomorphism

$$P[D] *_L Z_0 \longrightarrow X \setminus D$$

(iii) Suppose that $P[D] = P(X)$. Then the kernel L_0 of the action of L on Z_0 contains the commutator subgroup $[L, L]$.

For simplicity we denote $P(X)$ by P. Let us notice that in the theorem x_0 can be chosen so that $L \supset T$.

In the situation of the local structure theorem (iii), we see that the torus $A := L/L_0 = P/L_0 P_u$ is acting effectively on Z_0 (The group $L_0 P_u$ is denoted by P_0). And from $\mathbb{K}(X)^{(B)} = \mathbb{K}(Z_0)^{(B_L)} = \mathbb{K}(Z_0)^{(L)}$ one can identify $\Xi(A)$ with the group of characters

$$\Xi(X) = \{\chi \mid \mathbb{K}(X)_\chi^{(B)} \neq 0\}.$$

We shall write A_X (resp. $\mathfrak{a}_X$) if we want to stress a dependence on the variety X. Let us recall that as a corollary of the local structure theorem we get that general P_0-orbits coincide with general P_u-orbits.
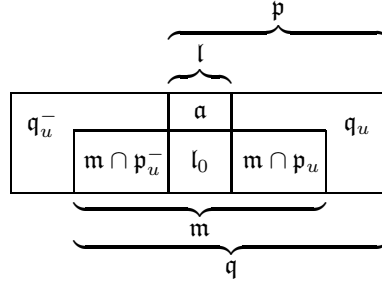
2. Equivariant maps to the flag varieties

To formulate a refined version of the local structure theorem we introduce some additional notation.

We denote by Q some parabolic subgroup of G containing P. Let M be the Levi subgroup of Q that contains the maximal torus T. We have a Levi decomposition $Q = Q_u \rtimes M$. Let us assume that $T \cap [M, M] \subset L_0$. Later in the proof of 4.1 we shall choose Q to be the stabilizer of the divisors of B-semi-invariant functions; then it satisfies this property.

It is easy to see that $Q_u \subset P_u$ and $L \subset M$. Consider the group $M_0 = [M, M]Z(L_0)$. We also put $Q_0 = Q_u \rtimes M_0$, so we have $A \cong M/M_0 \cong Q/Q_0$.

Let us embed $\mathfrak{a}$, the Lie algebra of A, into $\mathfrak{l}$ as the orthocomplement to $\mathfrak{l}_0$. Then the group $Z_G(\mathfrak{a})$ contains M. We can describe the relations among the introduced Lie algebras by the following picture taken from [17].

$$
\begin{array}{c}
\overbrace{\phantom{\mathfrak{q}_u^- \quad \mathfrak{a} \quad \mathfrak{q}_u}}^{\mathfrak{p}} \\
\overbrace{\phantom{\mathfrak{a}}}^{\mathfrak{l}} \\
\boxed{\mathfrak{q}_u^- \quad \boxed{\begin{array}{c} \mathfrak{a} \\ \boxed{\mathfrak{m} \cap \mathfrak{p}_u^-} \quad \mathfrak{l}_0 \quad \boxed{\mathfrak{m} \cap \mathfrak{p}_u} \end{array}} \quad \mathfrak{q}_u} \\
\underbrace{\phantom{\mathfrak{m} \cap \mathfrak{p}_u^- \quad \mathfrak{l}_0 \quad \mathfrak{m} \cap \mathfrak{p}_u}}_{\mathfrak{m}} \\
\underbrace{\phantom{\mathfrak{q}_u^- \quad \mathfrak{m} \quad \mathfrak{q}_u}}_{\mathfrak{q}}
\end{array}
$$

Identifying $\mathfrak{g} \cong \mathfrak{g}^*$ via the invariant bilinear form fixed in the conventions we see that $\mathfrak{l}, \mathfrak{l}_0, \mathfrak{m}, \mathfrak{m}_0, \mathfrak{a}$ are self-dual. Also we have $\mathfrak{p}_u = \mathfrak{p}^\perp \cong (\mathfrak{g}/\mathfrak{p})^* \cong (\mathfrak{p}_u^-)^*$, and $\mathfrak{q}_u = \mathfrak{q}^\perp \cong (\mathfrak{g}/\mathfrak{q})^* \cong (\mathfrak{q}_u^-)^*$. Let us denote

$$\mathfrak{a}^{pr} := \{\xi \in \mathfrak{a} | \ Z_G(\xi) = Z_G(\mathfrak{a}), \ g\xi \notin \mathfrak{a} \text{ for all } g \in G \setminus N_G(\mathfrak{a})\},$$

in fact $\mathfrak{a}^{pr}$ is obtained from $\mathfrak{a}$ by throwing away a finite union of hyperplanes.

We shall construct the morphism that is the main tool in the proof of the refined local structure theorem in the sense of Timashev.

Let us fix an effective G-linearized B-divisor D and the corresponding section σ_D with weight χ_D. Consider the action of M on the space of sections $H^0(X, \mathcal{O}(D))$. Let $V_\chi(M) := \langle M\sigma_D \rangle$ be the M-module generated by σ_D. It is simple since σ_D is B-semi-invariant. $V_\chi(M)$ can be considered as a simple Q-module fixed pointwise by Q_u. Indeed Q_u is a normal subgroup in Q that stabilizes the vector σ_D. Moreover $Z(M)$ acts by a character on the simple module $V_\chi(M)$.

Let $|V_\chi(M)|$ be the linear system on X (possibly not complete) corresponding to the Q-module $V_\chi(M) \subset H^0(X, \mathcal{O}(D))$.

Remark 2.1. The basepoint set of the linear system $|V_\chi(M)|$ is M-invariant and is equal to $\bigcap_{m \in M} mD$ since $V_\chi(M)$ is the linear span of the M-orbit of σ_D.

Consider the morphism:

$$\pi_D : X \setminus \bigcap_{m \in M} mD \longrightarrow \mathbb{P}(V_\chi(M)^*),$$

defined by the linear system $|V_\chi(M)|$. It is easy to see that π_D is Q-equivariant. This imply the following lemma:

Lemma 2.2. *Any orbit of the radical $Z(M) \ltimes Q_u$ of Q is contained in a fiber of π_D.*

Remark 2.3. If the divisor D is M-invariant, then $\mathbb{P}(V_\chi(M)^*)$ is a point. For our purposes it is sufficient to consider a divisor that is not M-stable. This condition implies that $\operatorname{codim} \bigcap_{m \in M} mD \geqslant 2$.

Now we are ready to state one of the main theorems of the paper.

Theorem 2.4. *Let D be an effective G-linearized B-divisor, with the canonical section $\sigma_D \in H^0(X, \mathcal{O}(D))_\chi^{(B)}$ and $P[D]$ be the stabilizer of D. Consider a parabolic subgroup Q of G containing P with a Levi subgroup M, that satisfies the inclusion $T \cap [M, M] \subset L_0$. Then the image of the morphism:*

$$\pi_D : \mathring{X} = X \setminus \left(\bigcap_{m \in M} mD \right) \longrightarrow \mathbb{P}(V_\chi(M)^*)$$

coincides with the flag variety $M\langle \sigma_{-\chi}^* \rangle \cong M/P_M^-$, where $P_M^- = M \cap P[D]^-$.

Proof. First let us recall that $M\langle \sigma_{-\chi}^* \rangle$ is the unique closed orbit in $\mathbb{P}(V_\chi(M)^*)$. We begin with the following well known lemmas.

Lemma 2.5. *Let $\lambda \in C_M^\circ$. Denote by $\operatorname{Ann} \sigma_\chi$ a hyperplane in $V_\chi(M)^*$ annihilating σ_χ. Then for any point $x \in \mathbb{P}(V_\chi(M)^*) \setminus \mathbb{P}(\operatorname{Ann} \sigma_\chi)$ we have $\lim_{t \to 0} \lambda(t)x = \langle \sigma_{-\chi}^* \rangle$, besides $Mx \nsubseteq \mathbb{P}(\operatorname{Ann} \sigma_\chi)$ for any $x \in \mathbb{P}(V_\chi(M)^*)$.*

Proof. Let $v \in V_\chi(M)^*$ be a vector representing x. Then $\langle Mv \rangle = V_\chi(M)^*$ by the simplicity of $V_\chi(M)^*$, in particular $Mx \nsubseteq \mathbb{P}(\operatorname{Ann} \sigma_\chi)$. For $v \in V_\chi(M)^* \setminus \operatorname{Ann} \sigma_\chi$ consider the decomposition in weight vectors:

$$v = c\sigma_{-\chi}^* + \sum_{\substack{\omega \in \operatorname{Wt}(V_\chi(M)^*) \\ \omega \neq -\chi}} c_\omega \sigma_\omega^*,$$

where $\sigma_\omega^* \in V_\chi(M)^*$ is a vector of weight ω and $c \neq 0$. Since $-\langle \lambda; \chi \rangle < \langle \lambda; \omega \rangle$, for $\lambda \in C_M^0$ and every $\omega \in \operatorname{Wt}(V_\chi(M)^*)$ which is distinct from $-\chi$, we obtain:

$$\lim_{t \to 0} \lambda(t)x = \lim_{t \to 0} \left\langle \sigma_{-\chi}^* + \sum_{\substack{\omega \in \operatorname{Wt}(V_\chi(M)^*) \\ \omega \neq -\chi}} t^{\langle \lambda, \chi + \omega \rangle} \frac{c_\omega}{c} \sigma_\omega^* \right\rangle = \langle \sigma_{-\chi}^* \rangle.$$

$\square$

Let us recall a following lemma.

Lemma 2.6. [16, Prop. 8.4.5] *Let G be a reductive group, T be a maximal torus and P be some parabolic subgroup containing T. Consider a one-parameter subgroup $\lambda \in \Lambda(T)$ such that $\langle \lambda, \alpha \rangle > 0$ for all $\alpha \in \Delta_{P_u}$. Then we have $\lim_{t \to 0} \lambda(t)p_u\lambda(t)^{-1} = e$ for $p_u \in P_u$.*

We note that by definition of π_D the image $\pi_D(X \setminus D)$ is equal to the complement of $\operatorname{Ann} \sigma_\chi$ in $\pi_D(\mathring{X})$. We also have $X \setminus D = \pi_D^{-1}(\mathbb{P}(V_\chi(M)^*) \setminus \mathbb{P}(\operatorname{Ann} \sigma_\chi))$. According to the Local Structure Theorem 1.5 we may choose a dense open B-invariant subset X° of $X \setminus D$ isomorphic to $P *_L Z_0$, where L_0 is acting trivially on Z_0. Let us choose a one-parameter subgroup λ of the torus $[M, M] \cap T$ such that $\lambda \in C_M^\circ$. Since $[M, M] \cap T \subset L_0$, the one-parameter subgroup λ acts trivially on Z_0. Let us calculate the limit $\lim_{t \to 0} \lambda(t)x$ for $x \in \pi_D(X^\circ)$ in two different ways. Since $\pi_D(X^\circ) \subset \mathbb{P}(V_\chi(M)^*) \setminus \mathbb{P}(\operatorname{Ann} \sigma_\chi)$ Lemma 2.5 implies the following:

Statement 2.7. *For $x \in \pi_D(X^\circ)$ we have $\lim_{t \to 0} \lambda(t)x = \langle \sigma_{-\chi}^* \rangle$.*

Let us calculate this limit in a different way. By the Local Structure Theorem the action of P_u is free on X° and we have $X^\circ = P_u Z_0$.

Statement 2.8. *Let λ be a one-parameter subgroup of the torus $[M, M] \cap T$ such that $\lambda \in C_M^\circ$. Then for $x = \pi_D(p_u z) \in \pi_D(X^\circ)$, where $p_u \in P_u$, $z \in Z_0$, we have $\lim_{t \to 0} \lambda(t) x = \pi_D(z)$.*

Proof. Consider the decomposition $Q = Q_u \rtimes M$; combining it with the inclusions $Q_u \subset P_u \subset Q$ we get $P_u = Q_u \rtimes (M \cap P_u)$. Thus we have $p_u = q_u m$, for $q_u \in Q_u$, $m \in M \cap P_u$. Thus we obtain

$$x = \pi_D(p_u z) = \pi_D(q_u m z) = m \pi_D(z).$$

Since $\lambda \in C_M^\circ$ is positive on all roots corresponding to $M \cap P_u$, by Lemma 2.6 we get $\lim_{t \to 0} \lambda(t) m \lambda(t)^{-1} = e$ for $m \in M \cap P_u$. The triviality of the action of λ on Z_0 implies that:

$$\lim_{t \to 0} \lambda(t) x = (\lim_{t \to 0} \lambda(t) m \lambda(t)^{-1}) \pi_D(z) = \pi_D(z) \in \pi_D(Z_0).$$

$\square$

Combining Statements 2.7 and 2.8 we get $\pi_D(Z_0) = \langle \sigma_{-\chi}^* \rangle$. Using the fact that $\pi_D(X^\circ) = (M \cap P_u) \pi_D(Z_0)$, we obtain that

$$M \pi_D(X^\circ) = M \pi_D(Z_0) = M \langle \sigma_{-\chi}^* \rangle.$$

Since $M \pi_D(X^\circ)$ is dense in $\pi_D(\mathring{X})$, this proves our theorem. $\square$

Corollary 2.9. *(of the proof of Theorem 1.9) Let X° be the dense open B-invariant subset of $X \setminus D$ isomorphic to $P *_L Z_0$, where L_0 is acting trivially on Z_0. The section Z_0 is contained in a fiber of the map π_D.*

Now we are able to describe the sets $X \setminus D$ and D as preimages for the map π_D of some subsets in M/P_M^-.

Proposition 2.10. *The set $X \setminus D$ is the preimage under π_D of the open Bruhat cell $B_M P_M^- / P_M^- \subset M/P_M^-$, where $B_M = B \cap M$. The set $\pi_D(D \cap \mathring{X})$ is equal to the preimage of the complement of this open cell in M/P_M^-.*

Proof. Let us recall that the complement of the open cell in $M \langle \sigma_{-\chi}^* \rangle \subset \mathbb{P}(V_\chi(M)^*)$ can be described as

$$\{\langle \sigma^* \rangle \in M/P_M^- \mid \langle \sigma^*, \sigma_\chi \rangle = 0\} \qquad (*)$$

We see that equality $\sigma_\chi(x) = 0$ (i.e. $x \in D$) is equivalent to $\langle \pi_D(x), \sigma_\chi \rangle = 0$. Thus we get that $X \setminus D$ maps into the open cell $B_M P_M^- / P_M^- \subset M/P_M^-$ and $D \cap \mathring{X}$ maps to the complement of the open cell. Since $\pi_D : \mathring{X} \to M/P_M^-$ is surjective this proves our proposition. $\square$

3. Relation between π_D and cross sections.

Now we state a result which relates the cross sections from the local structure theorem, introduced by Knop, and the fibers of π_D.

Proposition 3.1. *Consider an effective G-linearized B-divisor D with weight χ and the canonical section σ_χ. We have a map $\psi_D : X \setminus D \longrightarrow \mathfrak{g}^* : x \mapsto l_x$, where $l_x(\xi) = \frac{\xi \sigma_\chi}{\sigma_\chi}(x)$. For a point $x_0 \in X \setminus D$ consider $Z := \psi_D^{-1}(\psi_D(x_0))$. Let π_D be the map constructed in Theorem 2.4 for the divisor D. Then*

$$Z \subset \pi_D^{-1}(\pi_D(x_0))$$

Remark 3.2. This proposition is also valid for B-divisor D with a $P[D]$-regular weight χ_D. We omit a precise formulation and a proof for brevity.

Proof. We shall give two proofs of this proposition based on different observations.

Argument 1 Denote by $L[D]$ the Levi subgroup of $P[D]$ that is the stabilizer of $\psi_D(x_0)$. By Q-equivariance of π_D the translation of x_0 by an element from $P[D]_u$ preserves the condition $Z \subset \pi_D^{-1}(\pi_D(x_0))$; thus we can assume that $L[D] \supset L$. By the local structure theorem we have $X \setminus D = P[D] *_{L[D]} Z$. We can apply the local structure theorem to the action $L[D] : Z$ and to the divisor $D_0 \cap Z$ where $D_0 \subset X$ is a B-divisor with stabilizer $P := P(X)$. Thus we get that Z contains an open subset isomorphic to $(L[D] \cap P) *_L Z_0$ where the action $L_0 : Z_0$ is trivial. By Corollary 2.9 we know that $\pi_D(Z_0) = e(M \cap P[D]^-)$. Let us prove that the group $L[D] \cap P_u$ fixes this point. Indeed from the decomposition $P_u = (P_u \cap M)Q_u$ using the root decompositions with respect to T of the corresponding Lie algebras we get $L[D] \cap P_u = (L[D] \cap M \cap P_u)(L[D] \cap Q_u)$. The claim follows from the facts that Q_u-orbits lie in the fibers of π_D and that $L[D] \cap M$ fixes $e(M \cap P[D]^-)$. This implies that $\pi_D((L[D] \cap P_u)Z_0) = e(M \cap P[D]^-)$ and proves the proposition.

Argument 2 The set $\pi_D(X \setminus D)$ is equal to the open $P[D]_u \cap M$-orbit in $M/(M \cap P[D]^-)$. If the statement of the proposition was not true, there would exist a point $x \in Z$ such that $\pi_D(x) \neq \pi_D(x_0)$ and $\pi_D(x) = p\pi_D(x_0)$ for some $p \in P[D]_u \cap M$.

Consider the restriction map

$$\psi_D|_{\mathfrak{m}} : X \setminus D \longrightarrow \mathfrak{g}^* \longrightarrow \mathfrak{m}^*,$$

$$\psi_D|_{\mathfrak{m}} : x \mapsto l_x, \quad where \quad l_x(\xi) = \frac{\xi \sigma_\chi}{\sigma_\chi}(x), \ \forall \xi \in \mathfrak{m}.$$

The map ψ_D is $P[D]$-equivariant, and this implies $P[D] \cap M$-equivariance of $\psi_D|_{\mathfrak{m}}$. We have the following evident lemma.

Lemma 3.3. *The map $\psi_D|_{\mathfrak{m}} : X \setminus D \to \mathfrak{m}^*$ is equal to the composition of the map $\pi_D : \mathring{X} \to \mathbb{P}(V_\chi(M)^*)$ and the map $\mathbb{P}(V_\chi(M)^*) \setminus \mathbb{P}(\mathrm{Ann}\, \sigma_\chi) \to \mathfrak{m}^*$ defined as:*

$$\langle \sigma^* \rangle \mapsto l_{\langle \sigma^* \rangle}, \quad where \quad l_{\sigma^*}(\xi) = \frac{\langle \xi \sigma_\chi, \sigma^* \rangle}{\langle \sigma_\chi, \sigma^* \rangle}, \ \forall \xi \in \mathfrak{m}.$$

Applying Theorem 1.5 we obtain that the image of $\psi_D|_{\mathfrak{m}}(X \setminus D)$ is a single $P[D]_u \cap M$-orbit and the stabilizer of $\psi_D|_{\mathfrak{m}}(x_0)$ in $P[D]_u \cap M$ is trivial. By Lemma 3.3 the equality $\pi_D(x) = p\pi_D(x_0) = \pi_D(px_0)$ for $p \in P_u \cap M$ implies that $\psi_D|_{\mathfrak{m}}(x) = \psi_D|_{\mathfrak{m}}(px_0)$. Taking into account that

$$\psi_D|_{\mathfrak{m}}(x_0) = \psi_D|_{\mathfrak{m}}(x) = \psi_D|_{\mathfrak{m}}(px_0) = p\psi_D|_{\mathfrak{m}}(x_0),$$

we come to a contradiction since p does not stabilize $\psi_D|_{\mathfrak{m}}(x_0)$. $\square$

Proposition 3.4. *The fibers of the map π_D are irreducible. The set $\pi_D^{-1}(\pi_D(x_0))$ is identified with*

$$(L[D] \cap Q) \ltimes (P[D]_u \cap Q_u) *_{L[D] \cap Q} Z \cong (P[D]_u \cap Q_u) \times Z.$$

Proof. Let $x \in M/P_M^-$ be the point corresponding to the right coset eP_M^-. The preimage $\pi_D^{-1}((P[D]_u \cap M)x)$ of the open cell in M/P_M^- is irreducible, being the dense open subset $X \setminus D$ in X. It is isomorphic to $P[D]_u \cap M \times \pi_D^{-1}(x)$, which implies the irreducibility of $\pi_D^{-1}(x)$. Since the fibers of π_D are permuted transitively via the action of M this proves the first part of the proposition. The second part follows immediately from the isomorphism $X \setminus D \cong P[D]_u \cap M \times P[D]_u \cap Q_u \times Z$, inclusion $Z \subset \pi_D^{-1}(\pi_D(x_0))$ and the freeness of the action of $P[D]_u \cap M$ on M/P_M^-. $\square$

Consider the set of fundamental weights $\{\omega_\alpha\}$ of $[M, M]$ where $\alpha \in \Pi_M$.

Proposition 3.5. *Let α be a simple root in $\Pi_M \setminus \Pi_L$ and ω_α be the corresponding fundamental weight of M. Then there exists a unique prime B-divisor $D_\alpha \subset X$ such that the restriction of the weight χ_{D_α} to $[M,M] \cap T$ is equal to ω_α. Let D be a B-invariant divisor of weight χ_D such that $\chi_D|_{[M,M]\cap T} = \sum_{\alpha \in \Pi_M \setminus \Pi_L} n_\alpha \omega_\alpha$. Then $D = \sum_{\alpha \in \Pi_M \setminus \Pi_L} n_\alpha D_\alpha + D_0$ for some Q-invariant divisor D_0.*

Proof. Consider an effective B-invariant divisor D (see Lemma 1.4) such that $P[D] = P$ and let us denote $P_M = P \cap M$. For a divisor $F_\alpha = \overline{B_M s_\alpha P_M^-/P_M^-}$ (with the stabilizer P_α in M) let us take a natural number n such that nF_α is M-linearized. By Proposition 3.4 the B-divisor $D_\alpha := \pi_D^{-1}(F_\alpha)$ is irreducible. nD_α is the zero divisor of a section of the M-linearized line bundle $\pi_D^*(\mathcal{O}(nF_\alpha))$ which is a pullback the $B \cap M$-semi-invariant section σ_{nF_α}. We also have $\chi_{D_\alpha}|_{[M,M]\cap T} = \omega_\alpha$.

Let D be a prime B-invariant divisor that is not M-invariant. Since $\pi_D(D)$ is prime, by Proposition 2.10, it is equal to a prime Schubert divisor corresponding to some simple root $\alpha \in \Pi_M \setminus \Pi_L$. Thus $\chi_D|_{[M,M]\cap T}$ is equal to the restriction of the weight of this Schubert divisor which is ω_α.

To prove the uniqueness of D_α assume that we have two effective divisors D_α and D_α' such that the restrictions to $[M,M] \cap T$ of χ_{D_α} and $\chi_{D_\alpha'}$ are proportional to ω_α. This implies that the stabilizer in M of $D_\alpha + D_\alpha'$ is equal to P_α. Consider the map $\pi_\alpha := \pi_{D_\alpha + D_\alpha'}$. By Theorem 2.4 the image of π_α is equal to M/P_α^- and by Proposition 2.10 the divisor $D_\alpha + D_\alpha'$ maps to the prime divisor F_α which is the only B_M invariant divisor in M/P_α^-. This contradicts the irreducibility of the fibers of π_α and proves the first part of the corollary.

From the above we get a decomposition $D = \sum_{\alpha \in \Pi_M \setminus \Pi_L} m_\alpha D_\alpha + D_0$ for some Q-invariant D_0, and integer m_α. Comparing the $([M,M]\cap T)$-weights of both sides of this equality and using the equality $\chi_{D_\alpha}|_{[M,M]\cap T} = \omega_\alpha$, we get $\chi_D|_{[M,M]\cap T} = \sum_{\alpha \in \Pi_M \setminus \Pi_L} m_\alpha \omega_\alpha$. From the linear independence of fundamental weights we have $m_\alpha = n_\alpha = \langle \chi_D, \alpha^\vee \rangle$. $\square$

Corollary 3.6. *Let D be a B-divisor such that its weight χ_D has zero restriction to $[M,M] \cap T$. Then the divisor D is Q-invariant. In particular every B-semi-invariant rational function on X is Q-semi-invariant.*

4. Refined local structure theorem

Using Theorem 2.4 we shall derive a stronger version of the local structure theorem in the sense of Timashev [17, Thm.3]. Let us first introduce some notation.

We shall choose a parabolic subgroup $Q \supset P$ with Levi subroup M, for which $[M,M] \cap T \subset L_0$. Consider the set $\mathcal{E}$ of prime B-divisors that occur in the divisors (f) of rational B-semi-invariant functions $f \in k(X)^{(B)}$. In other words:

$$\mathcal{E} := \{E| \; \exists f \in k(X)^{(B)}, \; (f) = \sum a_D D, \; a_E \neq 0\}.$$

We denote by $Q \supset P$ the parabolic subgroup that is the stabilizer of all B-divisors from $\mathcal{E}$. Since Q stabilizes the divisor (f) for $f \in k(X)^{(B)}$, by a consequence of a theorem of Rosenlicht [19, Thm.3.1] we obtain that f is Q-semi-invariant. Let M be the Levi subgroup of Q containing the maximal torus T. We note that $T \cap [M,M] \subset L_0$, since $T \cap [M,M]$ is acting trivially on $f \in k(X)^{(B)}$. By Corollary 3.6 this parabolic subgroup Q is maximal in the set of parabolic subgroups stabilizing some B-divisor (in particular containing P) and satisfying the property $T \cap [M,M] \subset L_0$.

We choose an effective B-divisor D such that the stabilizer of D in Q is equal to P. Let us also choose an effective divisor $E \in \mathcal{E}$, such that Q is the stabilizer of E in G. In particular the stabilizer of $D \cup E$ is equal to P. Let $\sigma_E \in H^0(X, \mathcal{O}(E))^{(Q)}$ be the Q-semi-invariant section that defines E. Since E is effective the weight χ_E of

the divisor E is Q-regular. Let us denote by $X_1 := \mathring{X} \setminus E$ an open Q-invariant subset and by X° the P-invariant subset $X \setminus (D \cup E)$. We recall that $\mathring{X} = X \setminus \bigcap_{m \in M} mD$. We also consider the map:

$$\psi_E: \ X \setminus E \longrightarrow \mathfrak{g}^*: \ x \longrightarrow l_x(\xi) = \frac{\xi \sigma_E}{\sigma_E}(x).$$

Theorem 4.1. *There exists a point $x_0 \in X^\circ$ such that $M_{\pi_D(x_0)} = P_M^-$ and M is the stabilizer of $\psi_E(x_0)$. For the M-stable closed subset $Z_1 := \psi_E^{-1} \psi_E(x_0)$ of X_1 we have a Q-equivariant isomorphism*

$$Q *_M Z_1 \longrightarrow X_1.$$

Moreover $Z_1 \cong M/P_M^- \times Z_0$ for $Z_0 := \pi_D^{-1} \pi_D(x_0) \cap Z_1$. Here M acts on the product $M/P_M^- \times Z_0$ by left multiplication on M/P_M^- and via the quotient $M/M_0 \cong A$ on Z_0.

Proof. Consider a point $x_0 \in X^\circ$; since the Levi subgroups of Q are conjugate by the elements of Q_u, translating x_0 by the element of Q_u we may assume that M is the stabilizer of $\psi_E(x_0)$. By Proposition 2.10 X° is the preimage under π_D of the open Bruhat cell $(P_u \cap M)P_M^-/P_M^-$ in M/P_M^-. In particular translating x_0 by the element of $(P_u \cap M)$ we can assume that x_0 is in X°, the stabilizer of $\psi_E(x_0)$ is still M and $M_{\pi_D(x_0)} = P_M^-$.

The isomorphisms $Q *_M Z_1 \longrightarrow X_1$ and $Q *_M Z_1 \cong Q_u \times Z_1$ follows from the local structure theorem 1.5 applied to the point $x_0 \in X^\circ$, the divisor E and its stabilizer Q. From the M-equivariance of π_D and M-invariance of Z_1 we see that $Z_1 = MZ_0$. $\square$

Lemma 4.2. *We have the isomorphism $P *_L Z_0 \xrightarrow{\sim} X^\circ$, where the group L acts on Z_0 via the quotient L/L_0.*

Proof. Let us notice that $\pi_D(X^\circ) \supset \pi_D(Z_1 \cap X^\circ) \supset P_M \pi_D(x_0) \cong P_M/L$. This implies that the variety $Z_1 \cap X^\circ$ (which is equal to $Z_1 \setminus D$) projects P_M-equivariantly onto the open cell $\pi_D(X^\circ) = (P_u \cap M)P_M^-/P_M^-$ that is isomorphic to P_M/L, and in particular $Z_1 \setminus D \cong P_M *_L Z_0$.

The action of Q_u is free on X° and we have $X^\circ \cong Q_u \times (Z_1 \setminus D)$. Since $P = Q_u \rtimes P_M$ we have:

$$X^\circ \cong P *_{P_M} (Z_1 \setminus D) \cong P *_L Z_0.$$

Let us apply the local structure theorem to the effective divisor $D \cup E$ and the group P. We get that $X^\circ \cong P *_L Z_0'$ for some Z_0' with trivial action of L_0. The L-equivariant isomorphisms $X^\circ/P_u \cong Z_0' \cong Z_0$ imply the triviality of the action of L_0 on Z_0 which proves the lemma. $\square$

We shall prove the isomorphism $Z_1 \cong M/P_M^- \times Z_0$ in several steps. Let us first prove that the $P_u \cap M$-orbit of each point $z \in Z_1 \cap X^\circ$ is contained as a dense open subset in the orbit $M_0 z$. This was first noticed by Timashev ([17]) in less general settings for a general point of Z_1.

Step 1. Since the divisor of any B-semi-invariant rational function $f \in \mathbb{K}(X)^{(B)}$ is Q-semi-invariant, as noted before this function should be Q-semi-invariant as well. Thus we get $\mathbb{K}(X)^{(B)} = \mathbb{K}(X)^{(P)} = \mathbb{K}(X)^{(Q)}$. Since $\mathbb{K}(X)^U$ is generated by $\mathbb{K}(X)^{(B)}$, these equalities imply that $\mathbb{K}(X)^U = \mathbb{K}(X)^{P_0} = \mathbb{K}(X)^{Q_0}$. We shall prove that a general Q_0-orbit contains a general P_0-orbit as a dense open subset. Suppose the contrary. Then a general Q_0-orbit contains an infinite family of P_0-orbits. Due to the Rosenlicht Theorem general P_0-orbits are separated by rational invariants from $k(X)^{P_0}$. This implies that there exists an invariant $f \in k(X)^{P_0}$ which has

non-constant value on the family of P_0-orbits contained in a general Q_0-orbit, that contradicts the inclusion $f \in k(X)^{Q_0}$.

Step 2. The isomorphism $Q *_M Z_1 \cong Q_u \times Z_1 \longrightarrow X_1$ implies that for $z \in Z_1$ we have the following isomorphisms $Q_0 z \cong Q_u \times M_0 z$ and $P_0 z \cong Q_u \times (P \cap M_0) z$. From the above step and since $X^\circ = P_u Z_0 \cong P_u \times Z_0$, for a sufficiently general point $z \in Z_0$ we have $\overline{Q_0 z} = \overline{P_0 z}$. Thus for a sufficiently general point $z \in Z_0$ the previous equalities give

$$\overline{(P_u \cap M) z} = \overline{M_0 z}.$$

Step 3. Let us prove that $M_0 z \cong M_0 / P_{M_0}^- \cong M / P_M^-$ for $z \in Z_0$, where $P_{M_0}^- := P^- \cap M_0$.

Step 3a). First assume that $z \in Z_0$ is a sufficiently general point. Let us notice that $(P_u \cap M) z$ maps isomorphically to $(P_u \cap M) P_M^- / P_M^-$. The M-equivariance of π_D and the fact that π_D is an isomorphism on the dense subset $(P_u \cap M_0) z$ in $M_0 z$ imply that π_D maps $M_0 z$ isomorphically to M / P_M^-. In addition we have $(M_0)_z = (M_0)_{\pi_D(z)} = P_{M_0}^-$.

Step 3b). Consider an arbitrary point $z \in Z_0$. Let us recall that for a general point $z_0 \in Z_0$ we have $(M_0)_{z_0} = P_{M_0}^-$. Since the set of $P_{M_0}^-$-fixed points is closed in Z_0 we get the inclusion $(M_0)_z \supset P_{M_0}^-$. Using the M-equivariance of π_D we get the following sequence of inclusions:

$$P_{M_0}^- \subset (M_0)_z \subset (M_0)_{\pi_D(z)} = P_{M_0}^-.$$

The right and the left side of this sequence are equal, thus all the inclusions must be equalities, in particular $(M_0)_z = P_{M_0}^-$ for all $z \in Z_0$.

Since $Z_1 \cong M *_{P_M^-} Z_0$ and $P_{M_0}^-$ acts on Z_0 trivially we get $Z_1 \cong M / P_M^- \times Z_0$, where M acts on M / P_M^- by left multiplication and via the quotient $A = M / M_0 = P_M^- / P_{M_0}^-$ on Z_0. $\qquad\square$

To get the variant of the local structure theorem obtained by Timashev [17, Thm.3], we have to take the open subset of $Z_0 = \pi_D^{-1} \pi_D(x_0) \cap \psi_E^{-1} \psi_E(x_0)$ on which the A-action is free. Taking a smaller open subset of Z_0, we may suppose that it is isomorphic to $A \times C$, where the action on C is trivial.

Corollary 4.3. *(cf. [17, Thm.3]) There is a locally closed subset C of X such that the map*

$$Q *_M (M / P_M^- \times A \times C) \longrightarrow X$$

is an isomorphism to an open Q-invariant subset of X.

Remark 4.4. In Theorem 4.1 we can take any parabolic subgroup Q which is the stabilizer of some B-invariant divisor E and that satisfies the condition $T \cap [M, M] \subset L_0$. Corollary 3.6 states that such a parabolic subgroup Q stabilizes the divisors of all B-semiinvariant rational functions on X and the proof of Theorem 4.1 can be generalized to the case of such Q. However instead of Step 1 of the proof of Theorem 4.1, one can use the argument from Timashev [17, Claims 1,2] to prove that $\mathfrak{m}_0 x \subseteq \mathfrak{p}_u x$ for general $x \in X^\circ$. In particular that will imply that $\mathfrak{q}_0 x = \mathfrak{p}_u x$, as well as that $P_u x$ is open in $Q_0 x$. We shall reproduce the argument of Timashev for convenience of the reader.

Lemma 4.5. *(cf. [17, Claims 1,2]) $\mathfrak{m}_0 x = (\mathfrak{p}_u \cap \mathfrak{m}) x$ for sufficiently general $x \in Z_0$.*

Proof. From the local structure theorem we have $X^\circ = P *_L Z_0 \cong P_u \times Z_0$. Consider an open subset of Z_0, we can assume that it is isomorphic to $A \times C$, for some C with trivial T-action. Without loss of generality we may assume that $x \in C$. Suppose that $\mathfrak{m}_0 x \neq (\mathfrak{p}_u \cap \mathfrak{m}) x$. From the freeness of the P_u-action we get $\mathfrak{q}_0 x \neq \mathfrak{p}_u x$,

this implies the existence of $\xi_x \in T_x^* X$ such that $\xi_x|_{\mathfrak{p}_u x} = 0$ and $\xi_x|_{\mathfrak{q}_0 x} \neq 0$. In particular since $P_u x = P_0 x$ the first equality implies that $\mu_X(\xi_x) \in \mathfrak{p}_0^{\perp} = \mathfrak{a} + \mathfrak{p}_u$. Consider the L_0-module isomorphism $T_x^* X \cong (\mathfrak{a} + \mathfrak{p}_u) \oplus T_x^* C$; from $\xi_x|_{\mathfrak{p}_u x} = 0$ we also have $\xi_x \in \mathfrak{a} x \oplus T_x^* C$. In particular ξ_x is L_0-invariant. This implies that $\mu_X(\xi_x) \in (\mathfrak{a} + \mathfrak{p}_u)^{L_0} \subset \mathfrak{a} + \mathfrak{p}_u^{T \cap [M,M]} \subset \mathfrak{a} + \mathfrak{q}_u$ (where the last inclusion is due to the fact that $\mathfrak{p}_u \cap \mathfrak{m}$ is T-invariant but does not contain $T \cap [M, M]$-fixed vectors). Thus $\mu_X(\xi_x) \perp \mathfrak{q}_0$, contradicting the assumption $\xi_x|_{\mathfrak{q}_0 x} \neq 0$. $\square$

5. Families of nongeneric horospheres

In this section we shall construct a family of nongeneric horospheres. By horospheres we call the orbits of maximal unipotent subgroups of G. It will be proved that for the conormal bundle $\mathcal{N}^* X$ to some foliation of $\overline{U}$-orbits (for some maximal unipotent subgroup $\overline{U}$) constructed below, we have $\overline{G\mathcal{N}^* X} = T^* X$. The construction is based on ideas of F.Knop [12]. Our main idea to construct a Białynicki-Birula cell by means of special choice of a one-parameter subgroup that allows us to avoid using compactifications as in the cited paper of F.Knop. It also provides a deeper study of the constructed conormal bundle. This section is independent of the previous ones, so some notation will be slightly changed for brevity.

The crucial step is first to consider the case of a horospherical variety. Let us recall that a variety is called horospherical if the stabilizer of a general point contains a maximal unipotent subgroup. The case of general variety X will be studied by reducing to this one by horospherical contraction, whose definition and existence is stated below in Proposition 5.10. In a horospherical variety X one can find a G-invariant open subset isomorphic to $G/P_0^- \times C$, where C is supplied with the trivial action of G and $P_0^- = L_0 P_u^-$. Thus we shall construct a variety of degenerate horospheres for $X = G/P_0^-$ and extend it to a horospherical variety X by taking product with C. Let us introduce additional notation $M := Z_G(\mathfrak{a})$ and $M_0 := [M, M]Z(L_0)$. We note that M is not related with the group introduced in Section 2 and this notation retains to the end of the paper.

Proposition 5.1. *Let X be the horospherical variety G/P_0^-. Consider a Borel subgroup $\overline{B} \subset G$ with unipotent radical $\overline{U}$ such that there is an inclusion $\overline{\mathfrak{b}} \supset \mathfrak{a} + (\mathfrak{p}_u^- \cap \mathfrak{m})$ for its Lie algebra. Let $\mathcal{N}^*$ be the conormal bundle to the orbit $\overline{U} P_0^- / P_0^-$, then we have $\overline{G\mathcal{N}} = T^* X$.*

Proof. Let us remind that the cotangent bundle $T^* X$ is identified with $G *_{P_0^-} \mathfrak{p}_0^{-\perp} \cong G *_{P_0^-} (\mathfrak{a} + \mathfrak{p}_u^-)$. The fiber of conormal bundle to the orbit $\overline{U} P_0^- / P_0^-$ in the point $e P_0^-$ is identified with $(\overline{\mathfrak{u}} + \mathfrak{p}_0^-)^{\perp} = \overline{\mathfrak{u}}^{\perp} \cap \mathfrak{p}_0^{-\perp} = \overline{\mathfrak{b}} \cap (\mathfrak{a} + \mathfrak{p}_u^-) \supset \mathfrak{a} + (\mathfrak{p}_u^- \cap \mathfrak{m})$.

We need the following lemma.

Lemma 5.2. *Let P be some parabolic subgroup of G and L be a Levi subgroup of P. Then for a subalgebra $\mathfrak{a} \subseteq \mathfrak{z}(\mathfrak{l})$ we have an equality $\overline{P_u(\mathfrak{a} + (\mathfrak{p}_u \cap \mathfrak{m}))} = \mathfrak{a} + \mathfrak{p}_u$.*

Proof. We shall prove that the map $P_u \times (\mathfrak{a} + (\mathfrak{p}_u \cap \mathfrak{m})) \to \mathfrak{a} + \mathfrak{p}_u$ is dominant if we prove that its differential is surjective at the point (e, ξ), for general $\xi \in \mathfrak{a}$. Calculating the differential in (e, ξ) and using the equality $\mathfrak{p}_u = (\mathfrak{p}_u \cap \mathfrak{z}_{\mathfrak{g}}(\xi)) \oplus [\mathfrak{p}_u, \xi]$ for any $\xi \in \mathfrak{a}$ and the equality $\mathfrak{p}_u \cap \mathfrak{z}_{\mathfrak{g}}(\xi) = \mathfrak{p}_u \cap \mathfrak{m}$ which holds for general $\xi \in \mathfrak{a}$, we obtain that the differential:

$$\mathfrak{p}_u \times (\mathfrak{a} + (\mathfrak{p}_u \cap \mathfrak{m})) \to [\mathfrak{p}_u, \xi] + \mathfrak{a} + (\mathfrak{p}_u \cap \mathfrak{m}) = \mathfrak{a} + \mathfrak{p}_u$$

is indeed surjective. $\square$

The group P_u^- is acting on the fiber of T^*X over $x_0 = eP_0^-$ since it lies in stabilizer of this point. From the preceding lemma we obtain that

$$\overline{P_u^- \mathcal{N}_{x_0}^*} \supset \overline{P_u^- (\mathfrak{a} + (\mathfrak{p}_u^- \cap \mathfrak{m}))} = \mathfrak{a} + \mathfrak{p}_u^- = T_{x_0}^* X.$$

This implies that

$$\overline{G\mathcal{N}^*} = G(T_{x_0}^* X) = T^* X.$$

$\square$

We shall need the following elementary lemma.

Lemma 5.3. *Let X/T and Y/T be two families of equidimensional varieties over some variety T with a T-morphism $f : X/T \to Y/T$. Suppose there exists smooth points $t_0 \in T$ and $x_0 \in X_{t_0}$ such that the varieties X, X_{t_0} are smooth at $x_0 \in X_{t_0}$; Y, Y_{t_0} are smooth at $f(x_0)$, the map $f_{t_0} : X_{t_0} \to Y_{t_0}$ is a submersion at the point x_0 (i.e the map of tangent spaces $df_{t_0} : T_{x_0}X_{t_0} \to T_{f(x_0)}Y_{t_0}$ is surjective) and the projections $X \to T$, $Y \to T$ are submersions at x_0 and $f_{t_0}(x_0)$. Then the morphism $f_t : X_t \to Y_t$ is a submersion (and in particular is dominant) at a general point of X_t for a sufficiently general t.*

Before stating one of the main theorems of this section we recall the following proposition about adjoint orbits. Let us temporally change our notations only for this proposition.

Proposition 5.4. [4, §5.1, 5.5] *Consider an arbitrary parabolic subgroup P in G, a Levi subgroup L and the unipotent radical P_u. Let $\mathcal{O}_{\mathfrak{l}}$ be a nilpotent adjoint orbit of L in $\mathfrak{l}$. Let $x \in \mathfrak{z}(\mathfrak{l})$ be an arbitrary element of the center of $\mathfrak{l}$. There exists a unique G-orbit $\mathcal{O}_{\mathfrak{g}}$ meeting $x + \mathcal{O}_{\mathfrak{l}} + \mathfrak{p}_u$ in a dense open subset. The intersection $\mathcal{O}_{\mathfrak{g}} \cap (x + \mathcal{O}_{\mathfrak{l}} + \mathfrak{p}_u)$ is a single P-orbit. The following equality holds $\operatorname{codim}_{\mathfrak{g}} \mathcal{O}_{\mathfrak{g}} = \operatorname{codim}_{\mathfrak{l}} \mathcal{O}_{\mathfrak{l}}$. For a general point $z \in x + \mathcal{O}_{\mathfrak{l}} + \mathfrak{p}_u$ the stabilizer of z in $\mathfrak{p}_u^-$ is trivial and $[\mathfrak{p}_u^-, z]$ is transversal to $[\mathfrak{l}, z] + \mathfrak{p}_u$. We also have the equality for irreducible components $(P_z)^0 = (G_z)^0$.*

Now we are ready to deal with the case when X is an arbitrary G-variety. We are going to construct a family of $\overline{U}$-orbits for some maximal unipotent subgroup $\overline{U}$, such that the G-translate of the conormal bundle to this foliation is dense in T^*X.

Theorem 5.5. *Let X be a smooth G-variety. Consider the open subset $X^\circ \cong P *_L Z$ obtained by application of the Local Structure Theorem 1.5 to some effective B-divisor with stabilizer equal to the parabolic subgroup $P := P(X)$. Then there exists a maximal unipotent subgroup $\overline{U}$ with the following properties.*

(i) For any $z \in Z$ we have $\overline{U}z = (\overline{U} \cap U)z$.

(ii) Let $\mathcal{N}^*X$ be the conormal bundle to the foliation of orbits $\overline{U}z$ for $z \in Z$. Then we have $\overline{G\mathcal{N}^*X} = T^*X$.

Proof. To construct the desired family we proceed in several steps.

Step 1. Our aim is to construct a Białynicki-Birula cell with respect to a one parameter subgroup $\lambda(t) \subset Z(L_0)$. We shall choose λ in a special way. Let us recall that $\mathfrak{p} \cap \mathfrak{m}_0$ is a parabolic subalgebra of $\mathfrak{m}_0$ with a Levi subgroup $\mathfrak{l}_0$ and the unipotent radical $\mathfrak{p}_u \cap \mathfrak{m}$.

Let us take a one-parameter subgroup $\lambda : \mathbb{K}^\times \to T$ such that $\lambda(t) \in Z_{M_0}(L_0)$ and $\langle \lambda, \gamma \rangle < 0$ for all $\gamma \in \Delta_{(\mathfrak{p}_u \cap \mathfrak{m})}$.

Let us introduce the following groups: First consider $\overline{M} := Z_G(\lambda)$, with root system $\Delta_{\overline{M}} = \{\gamma \in \Delta | \langle \gamma; \lambda \rangle = 0\}$. It is a Levi subgroup of

$$\overline{Q} := \{g \in G| \text{ there exists a } \lim_{t \to 0} \lambda(t)g\lambda(t)^{-1} \text{ in } G\},$$

with Lie algebra

$$\overline{\mathfrak{q}} = \mathfrak{t} \oplus \bigoplus_{\langle \alpha, \lambda \rangle \geqslant 0} \mathfrak{g}_\alpha.$$

The unipotent radical of $\overline{Q}$ and the corresponding Lie algebra can be expressed by the formulae:

$$\overline{Q}_u = \{g \in G | \lim_{t \to 0} \lambda(t) g \lambda(t)^{-1} = e\} \qquad \overline{\mathfrak{q}}_u = \bigoplus_{\langle \alpha, \lambda \rangle > 0} \mathfrak{g}_\alpha.$$

In particular we have following the obvious inclusions $\overline{M} \supset L$ and $\overline{\mathfrak{q}}_u \supset (\mathfrak{p}_u^- \cap \mathfrak{m})$.

Let us fix an open P-invariant subset $X^\circ = P *_L Z$ of X constructed in the local structure theorem 1.5 applied to some effective B-divisor with stabilizer equal to the parabolic subgroup $P := P(X)$. We recall that L_0 acts trivially on Z. Consider the following open subset of Białynicki-Birula cell:

$$Z_\lambda := \{x \in X | \lim_{t \to 0} \lambda(t) x \in Z\},$$

It is well defined since $\lambda(t)$ fixes the points of Z. Let us define a map φ by the formula:

$$\varphi : Z_\lambda \to Z \quad \varphi(x) = \lim_{t \to 0} \lambda(t) x.$$

Let us show that $Z_\lambda \subset X^\circ$. Indeed if $\lim_{t \to 0} \lambda(t) x = z \in Z$, then the orbit of one-parameter subgroup λ of the point x intersects X° which is an open neighborhood of z. Thus X° being λ-invariant contains the whole $\lambda(t) x$.

Lemma 5.6. *For $x \in Z_\lambda$, $q \in \overline{Q}_u$ and $m \in \overline{M}$ we have the following equalities $\varphi(qx) = \varphi(x)$ and $\varphi(mx) = m\varphi(x)$.*

Proof. Indeed for $q \in \overline{Q}_u$ by definition $\lim_{t \to 0} \lambda(t) q \lambda(t)^{-1} = e$. Thus we get:

$$\varphi(qx) = \lim_{t \to 0} \lambda(t) q \lambda(t)^{-1} \cdot \lim_{t \to 0} \lambda(t) x = \lim_{t \to 0} \lambda(t) x = \varphi(x),$$

$$\varphi(mx) = \lim_{t \to 0} \lambda(t) m x = m \lim_{t \to 0} \lambda(t) x = \varphi(x)$$

$\square$

Proposition 5.7. *For $z \in Z$ we have $\varphi^{-1}(z) = (P_u \cap \overline{Q}_u) z$ and $Z_\lambda = (P_u \cap \overline{Q}_u) Z$.*

Proof. As was noticed before $Z_\lambda \subset X^\circ$. Let us write down the action of λ on the point $x \in X^\circ$, that we write in the form $x = p_u z$, for $p_u \in P_u$ and $z \in Z$. Since the action of λ is trivial on Z we get:

$$\lim_{t \to 0} \lambda(t)(p_u z) = \lim_{t \to 0} \lambda(t) p_u \lambda(t)^{-1} z$$

Taking into account that $\lambda(t) p_u \lambda(t)^{-1} \in P_u$ and the fact that the action of P_u is free on X° we get that $\lim_{t \to 0} \lambda(t) p_u \lambda(t)^{-1} z$ exists iff there exists the $\lim_{t \to 0} \lambda(t) p_u \lambda(t)^{-1}$. This implies that $p_u \in \overline{Q}$.

Let us prove that $p_u \in \overline{Q}_u$. Using the Levi decomposition $\overline{Q} = \overline{M} \ltimes \overline{Q}_u$ we get the decomposition $P_u \cap \overline{Q} = P_u \cap \overline{M} \ltimes P_u \cap \overline{Q}_u$, and in particular $p_u = m q_u$ for $m \in P_u \cap \overline{M}$ and $q_u \in P_u \cap \overline{Q}_u$. Hence we obtain:

$$\lim_{t \to 0} \lambda(t)(p_u z) = m \lim_{t \to 0} \lambda(t)(q_u z) = mz.$$

Thus the inclusion $mz \in Z$ is satisfied if and only if $m = e$, since $m \in P_u$, $X^\circ \cong P_u \times Z$. This gives the desired inclusion $p_u \in \overline{Q}_u$ and finishes the proof. $\square$

Consider the $\overline{Q}_u$-orbits of the points from Z. We shall prove that it is contained in the open subset X°.

Lemma 5.8. *For $z \in Z$ we have $(P_u \cap \overline{Q}_u) z = \overline{Q}_u z$.*

Proof. By Lemma 5.6 and Proposition 5.7 we get $\overline{Q}_u z \subseteq \varphi^{-1}(z) = (P_u \cap \overline{Q}_u)z$. That implies our lemma. $\qquad\square$

Step 2. Let us define the group $\overline{U}$. Consider the group $\overline{U}_{\overline{M}} = U \cap \overline{M}$. Let us define the required group as

$$\overline{U} = \overline{U}_{\overline{M}} \ltimes \overline{Q}_u \subset \overline{Q}.$$

It is a maximal unipotent subgroup of G being the preimage of a maximal unipotent subgroup in $\overline{M}$ under the morphism $\overline{Q} \to \overline{Q}/\overline{Q}_u \cong \overline{M}$.

Consider the family of orbits $\overline{U}z$ for $z \in Z$. From 5.8 we obtain that $\overline{U}z = \overline{U}_{\overline{M}}(\overline{Q}_u z) = \overline{U}_{\overline{M}}(\overline{Q}_u \cap P_u)z \subset P_u z$. This implies in particular that the orbits $\overline{U}z$ are contained in X° and that $\overline{U}z_1 \neq \overline{U}z_2$ for $z_1 \neq z_2$ (since $P_u z_1 \cap P_u z_2 = \emptyset$).

Lemma 5.9. *The orbit $\overline{U}z$ for $z \in Z$ is stable under the group*

$$\overline{S} := (L_0 \ltimes (\overline{M} \cap P_u)) \ltimes \overline{Q}_u \subset \overline{Q}.$$

Proof. We need to prove that L_0 normalizes $\overline{U}z$. But this follows from

$$L_0 \overline{U}z = L_0(\overline{M} \cap P_u)\overline{Q}_u z = (\overline{M} \cap P_u)\overline{Q}_u L_0 z = \overline{U}z,$$

where we have used that $L_0 \subset G_z$. $\qquad\square$

Now we are ready to prove part *(ii)* of the theorem. We shall give two arguments one is based on degeneration to horospherical variety the other is based on the calculation of the image of the moment map[2]. Let us recall the definition of conormal bundle to the foliation of $\overline{U}$-orbits:

$$\mathcal{N}^* X = \{(x, \xi) \in T^* X \mid x \in \overline{U}Z, \quad \langle \overline{\mathfrak{u}}x, \xi \rangle = 0\}$$

Argument 1: By [15] (see also [9]) we know that every G-variety X admits a degeneration to a horospherical variety.

Proposition 5.10. *For a G-variety X there exists a $G \times \mathbb{K}^\times$-variety $\mathcal{X}$ and a surjective G-invariant morphism $\tau : \mathcal{X} \to \mathbb{A}^1$ that is equivariant with respect to the action $\mathbb{K}^\times : \mathbb{A}^1$, such that*

> *(i) For $t \neq 0$ the fiber $X_t := \tau^{-1}(t)$ is isomorphic to X. The fiber X_0 is a smooth horospherical variety, shrinking X we may assume that $X_0 \cong G/P_0^- \times C$ (for some variety C).*
> *(ii) The morphism τ is equidimensional and flat. By shrinking X and $\mathcal{X}$, we can assume that $\mathcal{X}$ and τ are smooth.*
> *(iii) For the fibers of τ we have $P(X_t) = P$, the group L_0 is independent of t and in particular $\mathfrak{a}_{X_t} = \mathfrak{a}$.*

Let us choose a B-invariant divisor $D \subset X$ with stabilizer $P := P(X)$. We extend this divisor to a $B \times \mathbb{K}^\times$–divisor $\mathcal{D}$ on the $G \times \mathbb{K}^\times$–variety $\mathcal{X}$ in the following way. Using the isomorphism $\mathcal{X} \setminus X_0 \cong X \times \mathbb{K}^\times$ we extend D to a $B \times \mathbb{K}^\times$–divisor on $\mathcal{X} \setminus X_0$. We are finished by setting $\mathcal{D}$ to be the closure of this divisor.

Having constructed the $B \times \mathbb{K}^\times$–divisor $\mathcal{D}$, we see, that the $P[D]$-regularity condition for the weight χ_D of the section σ_D is the same as the $P[\mathcal{D}]$-regularity condition of the weight $\chi_{\mathcal{D}}$ of the section $\sigma_{\mathcal{D}}$. So we can apply the local structure theorem to get the following statement.

Statement 5.11. *Consider the map $\psi_{\mathcal{D}} : \mathcal{X} \to \mathfrak{g}^*$ constructed in the local structure theorem. Then $\operatorname{Im} \psi_{\mathcal{D}} = \chi_D + \mathfrak{p}_u$. Defining $\mathcal{Z} := \psi_{\mathcal{D}}^{-1}(\chi_D)$, we have*

$$\mathcal{X} \setminus \mathcal{D} \cong P *_L \mathcal{Z}.$$

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[2]It is generalization of F.Knop's proof [10] for nondegenerate varieties

Let X be an affine variety and $D = (f)$ for some $f \in \mathbb{K}[X]^{(B)}$ or X be a projective variety and D be a pullback of some B-invariant hyperplane section, then $\mathcal{Z} \cong Z \times \mathbb{A}^1$.

Proof. Let X be an affine variety. We are left to prove that $\mathcal{Z} \cong Z \times \mathbb{A}^1$. Let us recall that $\mathcal{X} /\!\!/ U \cong X /\!\!/ U \times \mathbb{A}^1$ ([15, Prop. 11]). Then $\mathcal{D}$ is the divisor of zeroes of $F \in \mathbb{K}[\mathcal{X}]^B$, that is a pullback of f under projection $\mathcal{X} /\!\!/ U \to X /\!\!/ U$. The desired equality follows from:

$$\mathbb{K}[\mathcal{Z}] = \mathbb{K}[\mathcal{X} \setminus \mathcal{D}]^U = K[\mathcal{X}]^U_F = K[X]^U_f \otimes \mathbb{K}[t] = K[Z] \otimes \mathbb{K}[t],$$

where $K[\mathcal{X}]^U_F, K[X]^U_f$ are the localizations with respect to F and f correspondingly.

Taking an affine cone $\widehat{X}$ over X we reduce a projective case to an affine (we recall from [9] that for projective X, $\mathcal{X}$ is constructed by applying a standard construction of degeneration [15] to $\widehat{X}$ and taking a quotient by dilatations). $\qquad\square$

The family of orbits constructed in the second step can be extended to the whole variety $\mathcal{X}$. Since $P(X) = P(X_0)$ and $\mathfrak{a}_X = \mathfrak{a}_{X_0}$ we can choose the same λ as in Step 1 for the variety $\mathcal{X}$ and all its fibers X_t. We define a Białynicki-Birula cell for $\mathcal{X}$:

$$\mathcal{Z}_\lambda = \{x \in \mathcal{X} | \ \exists \ \lim_{t \to 0} \lambda(t)x \in \mathcal{Z}\}.$$

Applying Proposition 5.7 we get

$$\mathcal{Z}_\lambda = (P_u \cap \overline{Q}_u) \times \mathcal{Z}.$$

Consider the conormal bundle $\mathcal{N}X$ to the constructed foliation of $\overline{U}$-orbits. It fits into the following family of the conormal bundles to the foliations of $\overline{U}$-orbits in the fibers of $\tau : \mathcal{X} \to \mathbb{A}^1$:

$$\mathcal{N}^*\mathcal{X} = \{(x, t, \xi) \in T^*\mathcal{X}/\tau^*(T^*\mathbb{A}^1) | \ x \in \overline{U}\mathcal{Z}_t, \quad \langle \overline{u}x, \xi \rangle = 0\}$$

We note that the restriction of $T^*\mathcal{X}/\tau^*(T^*\mathbb{A}^1)$ to X_t is isomorphic to T^*X_t. From Proposition 5.1 we know that the map $G \times \mathcal{N}^*X_0 \to T^*X_0$ is dominant. This implies (by application of Lemma 5.3) that $G \times \mathcal{N}^*X_t \to T^*X_t$ is dominant for general t, which proves our claim.

Argument 2: The second proof is based on the study of the image of the conormal bundle $\mathcal{N}^*X$ under the moment map μ_X. Since $\overline{S}$ normalizes the orbits $\overline{U}z$ (for $z \in Z$), we get

$$\mathcal{N}^*X = \{(x, \xi) \in T^*X | \ x \in \overline{U}Z, \quad \langle \overline{s}x, \xi \rangle = 0\}$$
$$\mu_X(\mathcal{N}^*X) \subset \overline{\mathfrak{s}}^\perp = \mathfrak{a} + (\mathfrak{p}_u \cap \overline{\mathfrak{m}}) + \overline{\mathfrak{q}}_u \supset \mathfrak{a} + (\mathfrak{p}^-_u \cap \mathfrak{m}).$$

Remark 5.12. By the construction of $\overline{\mathfrak{q}}$ we have the inclusion $\overline{\mathfrak{q}}_u \supset (\mathfrak{p}^-_u \cap \mathfrak{m})$ and the equality $\mathfrak{a} + (\mathfrak{p}^-_u \cap \mathfrak{m}) = \mathfrak{a} + (\mathfrak{p}_u \cap \overline{\mathfrak{m}} + \overline{\mathfrak{q}}_u) \cap \mathfrak{m}$.

Let us denote $\overline{P} = N_G(\overline{S})$. We have $\overline{\mathfrak{p}}_u = \mathfrak{p}_u \cap \overline{\mathfrak{m}} + \overline{\mathfrak{q}}_u$. The next proposition gives us information about the image $\mu_X(\mathcal{N}_z X)$.

Proposition 5.13. *Let $\mathcal{N}_z X$ be the fiber of $\mathcal{N}X$ over $z \in Z$. Consider the T-equivariant projection of $\mathfrak{a} + \overline{\mathfrak{p}}_u$ to the subspace $\mathfrak{a} + \overline{\mathfrak{q}}_u \cap \mathfrak{p}^-_u$ with the fibers parallel to the subspace $\overline{\mathfrak{q}} \cap \mathfrak{p}_u$. Then the image of $\mu_X(\mathcal{N}_z X)$ under this projection is equal to $\mathfrak{a} + \overline{\mathfrak{q}}_u \cap \mathfrak{p}^-_u$.*

Proof. Consider the T-stable decomposition $\mathfrak{g} = \mathfrak{a} + (\overline{\mathfrak{q}}_u \cap \mathfrak{p}^-_u + \overline{\mathfrak{q}}^-_u \cap \mathfrak{p}_u) + \mathfrak{g}_0$, where $\mathfrak{g}_0$ is orthogonal to the other direct summands. The restriction of the pairing to $\overline{\mathfrak{q}}_u \cap \mathfrak{p}^-_u + \overline{\mathfrak{q}}^-_u \cap \mathfrak{p}_u$ is non-degenerate and the subspaces $\overline{\mathfrak{q}}_u \cap \mathfrak{p}^-_u, \overline{\mathfrak{q}}^-_u \cap \mathfrak{p}_u$ are isotropic. Moreover $\overline{\mathfrak{q}} \cap \mathfrak{p}_u \subset \mathfrak{g}_0$, and the elements of $\mathfrak{a} + (\overline{\mathfrak{q}}_u \cap \mathfrak{p}^-_u)$ are identified with the linear functions on $\mathfrak{a} + (\overline{\mathfrak{q}}^-_u \cap \mathfrak{p}_u)$. From the inclusion

$$T_z X \supset \overline{u}z \oplus (\mathfrak{a} + \overline{\mathfrak{q}}^-_u \cap \mathfrak{p}_u)z = (\overline{\mathfrak{q}} \cap \mathfrak{p}_u)z \oplus (\mathfrak{a} + \overline{\mathfrak{q}}^-_u \cap \mathfrak{p}_u)z,$$

we see that any linear function η on $\mathfrak{a} + (\overline{\mathfrak{q}}_u^- \cap \mathfrak{p}_u)$ can be lifted to an element $\xi \in T_z^* X$ that is zero on $(\overline{\mathfrak{q}} \cap \mathfrak{p}_u)z = \overline{\mathfrak{u}}z$. We found $\xi \in \mathcal{N}_z^* X$ such that the projection of $\mu_X(\xi)$ to $\mathfrak{a} + \overline{\mathfrak{q}}_u \cap \mathfrak{p}_u^-$ is equal to η. This proves the proposition. $\square$

Remark 5.14. Consider some point $z \in Z$. The one-parameter subgroup λ acts on $T_z X$ since z is fixed by λ. In the proof of Proposition 5.13 we constructed a subspace $V_z \subset \mathcal{N}_z^* X$ such that $\mu_X(V_z)$ maps isomorphically to $\mathfrak{a} + \overline{\mathfrak{q}}_u \cap \mathfrak{p}_u^-$ under the T-equivariant projection. Let us notice that the decomposition

$$T_z X = (\overline{\mathfrak{q}} \cap \mathfrak{p}_u)z \oplus (\mathfrak{a} + \overline{\mathfrak{q}}_u^- \cap \mathfrak{p}_u)z \oplus R$$

can be taken λ-equivariant this implies that V_z can be chosen λ-invariant.

Proposition 5.15. *We have the equality* $\overline{\mu_X(\mathcal{N}^*X)} = \mathfrak{a} + \overline{\mathfrak{p}}_u = \mathfrak{a} + \mathfrak{p}_u \cap \overline{\mathfrak{m}} + \overline{\mathfrak{q}}_u$.

Proof. Since $\overline{P}$ normalizes the foliation of orbits it also normalizes $\mathcal{N}^* X$. Thus to prove the proposition it is sufficient to show that $\overline{P}\mu_X(\mathcal{N}_z X)$ is dense in $\overline{\mathfrak{s}}^\perp = \mathfrak{a} + \overline{\mathfrak{p}}_u$. We shall use the following lemma.

Lemma 5.16. *Consider the action of* $P_u \cap \overline{Q}$ *on* $\mathfrak{a} + \overline{\mathfrak{p}}_u$. *Let* $\xi \in \mathfrak{a} + \overline{\mathfrak{p}}_u$ *be a general point. Then the stabilizer of* ξ *in* $P_u \cap \overline{Q}$ *is trivial. Moreover if* $\xi \in \mathfrak{a}^{pr} + \mathfrak{p}_u \cap \overline{\mathfrak{q}}$ *then* $(P_u \cap \overline{Q})\xi = \xi + \mathfrak{p}_u \cap \overline{\mathfrak{q}}$.

Proof. The first claim follows from the second one. Let us take $\xi \in \mathfrak{a}^{pr}$. We have the inclusion $(P_u \cap \overline{Q})\xi \subset \xi + \mathfrak{p}_u \cap \overline{\mathfrak{q}}$ (it follows from the equality $P_u \cap \overline{Q} = \exp(\mathfrak{p}_u \cap \overline{\mathfrak{q}})$ and the inclusion $[\xi, \mathfrak{p}_u \cap \overline{\mathfrak{q}}] \subset \mathfrak{p}_u \cap \overline{\mathfrak{q}}$). From $\mathfrak{p}_u \cap \mathfrak{z}_\mathfrak{g}(\xi) = (\mathfrak{p}_u \cap \mathfrak{m}) \subset \overline{\mathfrak{q}}_u^-$ we get that the stabilizer of ξ in $P_u \cap \overline{Q}$ is trivial and we have the equality $[\xi, \mathfrak{p}_u \cap \overline{\mathfrak{q}}] = \mathfrak{p}_u \cap \overline{\mathfrak{q}}$. This implies that the tangent space in ξ to the orbit $(P_u \cap \overline{Q})\xi$ coincides with $\mathfrak{p}_u \cap \overline{\mathfrak{q}}$. Thus $(P_u \cap \overline{Q})\xi$ is dense in $\xi + \mathfrak{p}_u \cap \overline{\mathfrak{q}}$. Since any orbit of a unipotent group in an affine variety is closed we get $(P_u \cap \overline{Q})\xi = \xi + \mathfrak{p}_u \cap \overline{\mathfrak{q}}$. $\square$

By Proposition 5.13 we get that there exists $\xi = \xi_0 + \xi_+ \in \mu_X(\mathcal{N}_z^* X)$ where $\xi_0 \in \mathfrak{a}^{pr}$ and $\xi_+ \in \mathfrak{p}_u \cap \overline{\mathfrak{q}}$. By Lemma 5.16 we get that the stabilizer of ξ in $\mathfrak{p}_u \cap \overline{\mathfrak{q}}$ is trivial and $[\mathfrak{p}_u \cap \overline{\mathfrak{q}}, \xi] = \mathfrak{p}_u \cap \overline{\mathfrak{q}}$. Since the projection of $\mu_X(\mathcal{N}_z X)$ to the subspace $\mathfrak{a} + \mathfrak{p}_u^- \cap \overline{\mathfrak{q}}_u$ is surjective we get that $\mu_X(\mathcal{N}_z^* X) + \mathfrak{p}_u \cap \overline{\mathfrak{q}} = \overline{\mathfrak{s}}^\perp$. Calculating the differential of the map $(P_u \cap \overline{Q}) \times \mu_X(\mathcal{N}_z^* X) \to (P_u \cap \overline{Q})\mu_X(\mathcal{N}_z^* X)$ in the point ξ we get

$$(\mathfrak{p}_u \cap \overline{\mathfrak{q}}) \times \mu_X(\mathcal{N}_z^* X) \to [\mathfrak{p}_u \cap \overline{\mathfrak{q}}, \xi] + \mu_X(\mathcal{N}_z^* X) = \overline{\mathfrak{s}}^\perp.$$

This implies that $(P_u \cap \overline{Q})\mu_X(\mathcal{N}_z^* X)$ is dense in $\overline{\mathfrak{s}}^\perp$ and proves the proposition. $\square$

Remark 5.17. The Proposition 5.15 can also be proved in two different ways: via the construction of a family of linear subspaces that tends to $\mathfrak{a} + \overline{\mathfrak{q}}_u \cap \mathfrak{p}_u^-$ or via degeneration to a horospherical variety.

A second proof of Proposition 5.15. Assume for a moment that we have constructed a one-parameter family of linear subspaces $V_t \subset \overline{\mathfrak{s}}^\perp$ such that $V_t \subset \mu_X(\mathcal{N}_{z_t}^* X)$ for some $z_t \in Z$, for any $t \neq 0$, and $V_0 = \mathfrak{a} + \mathfrak{p}_u^- \cap \overline{\mathfrak{q}}_u$. By Lemma 5.2 $\overline{P}_u V_0$ which contains $\overline{P}_u(\mathfrak{a} + (\mathfrak{p}_u^- \cap \mathfrak{m}))$ is dense in $\mathfrak{a} + \overline{\mathfrak{p}}_u = \overline{\mathfrak{s}}^\perp$. Applying Lemma 5.3 to the map $f_t : \overline{P}_u \times V_t \to \overline{\mathfrak{s}}^\perp$ and using that f_0 is dominant, we get that f_t is dominant for almost all t. The fact that $\overline{P}_u V_t$ is dense in $\overline{\mathfrak{s}}^\perp$ and the following chain of inclusions proves the proposition:

$$\mu_X(\mathcal{N}^*X) \supset \overline{P}_u \mu_X(\mathcal{N}_{z_t}^* X) \supset \overline{P}_u V_t.$$

To construct the desired family of subspaces let us fix a strictly dominant one-parameter subgroup $\lambda_0 : \mathbb{K}^* \to T$. By Proposition 5.13 the projection of $\mu_X(\mathcal{N}_z^* X)$

for $z \in Z$ to $\mathfrak{a} + \mathfrak{p}_u^- \cap \overline{\mathfrak{q}}_u$ is surjective, thus we can fix a subspace $V_1 \subset \mu_X(\mathcal{N}_z^* X)$ that maps isomorphically to $\mathfrak{a} + \mathfrak{p}_u^- \cap \overline{\mathfrak{q}}_u$.

For the desired family let us take the closure of the family $V_t := \lambda_0(t) V_1 \subset \mu_X(\mathcal{N}_{\lambda_0(t)z}^* X)$ in the Grassmannian of $\dim V_1$ vector subspaces of $\mathfrak{g}$. By next lemma (that is the characterization of the open Schubert cell as the Białynicki-Birula cell) we get that in this Grassmannian we have the following limit $\langle V_0 \rangle := \lim_{t \to 0} \langle V_t \rangle = \langle \mathfrak{a} + \mathfrak{p}_u^- \cap \overline{\mathfrak{q}}_u \rangle$.

Lemma 5.18. [16, Prop. 8.5.1] *Let $\lambda(t)$ be a one-dimensional torus acting on a linear space V. Let $V_{\leqslant 0}$ be the sum of components of V with nonpositive λ-weights, and $\pi_{\leqslant 0}$ be the corresponding λ-equivariant projection. Consider the subspace $W \subset V$ that maps isomorphically to $V_{\leqslant 0}$ under the projection $\pi_{\leqslant 0}$. Then we have the following limit in the Grassmannian:*

$$\lim_{t \to 0} \lambda(t) \langle W \rangle = \langle V_{\leqslant 0} \rangle.$$

$\square$

Third proof of Proposition 5.15. Let us include $\mathcal{N}^* X$ into the family $\mathcal{N}^* \mathfrak{X}$ as we have done in Argument 1. We also know that $\mu_{X_t}(\mathcal{N}^* X_t) \subset \mathfrak{a} + \overline{\mathfrak{p}}_u$. We can do explicit calculations of the moment map for horospherical variety $X_0 \cong G/P_0^- \times C$. Indeed the image $\mu_X(\mathcal{N}_{x_0}^* X_0)$ for $x_0 = eP_0^-$ is identified with

$$(\overline{\mathfrak{u}} + \mathfrak{p}_0^-)^\perp = \overline{\mathfrak{b}} \cap (\mathfrak{a} + \mathfrak{p}_u^-) = \mathfrak{a} + \mathfrak{p}_u^- \cap \overline{\mathfrak{q}}_u \supset \mathfrak{a} + (\mathfrak{p}_u^- \cap \mathfrak{m}) = \mathfrak{a} + (\overline{\mathfrak{p}}_u \cap \mathfrak{m}).$$

The following inclusions for a horospherical variety X_0:

$$\mu_X(\mathcal{N}^* X_0) \supset \overline{P}_u \mu_X(\mathcal{N}_{x_0}^* X_0) \supset \overline{P}_u(\mathfrak{a} + (\overline{\mathfrak{p}}_u \cap \mathfrak{m}))$$

and the density of $\overline{P}_u(\mathfrak{a} + (\overline{\mathfrak{p}}_u \cap \mathfrak{m}))$ in $\mathfrak{a} + \overline{\mathfrak{p}}_u$ (see Lemma 5.2) imply that $\overline{\mu_X(\mathcal{N}^* X_0)} = \mathfrak{a} + \overline{\mathfrak{p}}_u$.

Applying Lemma 5.3 to the triple of varieties $(\mathcal{N}^* \mathfrak{X}, (\mathfrak{a} + \overline{\mathfrak{p}}_u) \times \mathbb{A}^1, \mathbb{A}^1)$ and the map $\mu_{\mathfrak{X}} = (\mu_{X_t}, t) : \mathcal{N}^* X_t \to (\mathfrak{a} + \overline{\mathfrak{p}}_u) \times \mathbb{A}^1$ and using the equality $\overline{\mu_X(\mathcal{N}^* X_0)} = \mathfrak{a} + \overline{\mathfrak{p}}_u$ we get that for a general $t \in \mathbb{A}^1$ we have $\overline{\mu_X(\mathcal{N}^* X_t)} = \mathfrak{a} + \overline{\mathfrak{p}}_u$. This proves the proposition. $\square$

We shall be finished after proving next proposition

Proposition 5.19. *The variety $\overline{P}_u^- \mathcal{N}^* X$ is dense in $T^* X$.*

Proof. By considering the differential of the map $\overline{P}_u^- \times \mathcal{N}^* X \to \overline{P}_u^- \mathcal{N}^* X$ we see that it is sufficient to prove that for some $\alpha \in \mathcal{N}^* X$ the tangent space to $T^* X$ in the point α is equal to the sum of $\overline{\mathfrak{p}}_u^- \alpha$ and the tangent space to $\mathcal{N}^* X$. Let us take α such that $\xi = \mu_X(\alpha) \in \mathfrak{a} + \overline{\mathfrak{p}}_u$ is a sufficiently general point. Proposition 5.4 implies that that $d\mu_X$ maps isomorphically $\overline{\mathfrak{p}}_u^- \alpha$ onto $\overline{\mathfrak{p}}_u^- \xi \cong \overline{\mathfrak{p}}_u^-$ and that $\overline{\mathfrak{p}}_u^- \xi$ is transversal to $\mathfrak{a} + \overline{\mathfrak{p}}_u = \overline{\mu_X(\mathcal{N}^* X)}$ in the point ξ. The intersection of the tangent space to $\mu_X(\mathcal{N}^* X)$ in the point ξ with the subspace $d\mu_X(\overline{\mathfrak{p}}_u^- \alpha)$ is equal to $(\mathfrak{a} + \overline{\mathfrak{p}}_u) \cap \overline{\mathfrak{p}}_u^- \xi = 0$. This implies that $\overline{\mathfrak{p}}_u^- \alpha$ is transversal to $\mathcal{N}^* X$ in α. The transversality of $\overline{\mathfrak{p}}_u^- \alpha$ and $\mathcal{N}^* X$ combined with the equality $\mathrm{codim}_{T^* X} \mathcal{N}^* X = \dim P_u = \dim \overline{P}_u$ implies our claim. $\square$

$\square$

Corollary 5.20. *The closure of the image of the moment map is equal to $\overline{\mu_X(T^* X)} = G(\mathfrak{a} + \overline{\mathfrak{p}}_u) = G(\mathfrak{a} + \mathfrak{p}_u^-)$.*

Proof. We have $\overline{\mu_X(T^* X)} = \overline{G \mu_X(\mathcal{N}^* X)} = G(\mathfrak{a} + \overline{\mathfrak{p}}_u)$. The equality $G(\mathfrak{a} + \overline{\mathfrak{p}}_u) = G(\mathfrak{a} + \mathfrak{p}_u^-)$ follows from the Lemma 5.2 and the equality $(\overline{\mathfrak{p}}_u \cap \mathfrak{m}) = (\mathfrak{p}_u^- \cap \mathfrak{m})$. $\square$

The following lemma states that for the considered family of $\overline{U}$-orbits parameterized by Z there exists a normalizer of general position.

Lemma 5.21. *For a general point $z \in Z$ the normalizer of the orbit $\overline{U}z$ is equal to some fixed group $\widetilde{S}$ normalized by T.*

Proof. For $z \in Z$ let $\widetilde{S}$ be the normalizer of $\overline{U}z$. Since $N_G(\widetilde{S})$ is a parabolic subgroup of G it also contains $N_G(\overline{U})$ which is a unique Borel subgroup containing $\overline{U}$; thus $T \subset N_G(\widetilde{S})$. Since $Z \cap \overline{U}z = z$ we have $N_G(\widetilde{S}) \cap T = T_z = L_0 \cap T$. The number of horospherical subgroups normalized by T and such that $\widetilde{S} \cap T = T_0$ is finite, thus for a general point of Z the normalizer of $\overline{U}z$ is equal to some fixed horospherical subgroup $\widetilde{S}$. $\qquad\square$

Let us shrink Z in such a way that for any $z \in Z$ the normalizer of the orbit $\overline{U}z$ is equal to $\widetilde{S}$. We can define the normalizer $\widetilde{P}$ of the family of orbits $\overline{U}z$ for $z \in Z$, i.e. the group that consists of $p \in \widetilde{P}$ such that for any $z_1 \in Z$ there exists $z_2 \in Z$ such that $p\overline{U}z_1 = \overline{U}z_2$.

Theorem 5.22. *The normalizer of the orbit $\overline{U}z$ for $z \in Z$ is equal to $\overline{S}$. The equality $g\overline{U}z = \overline{U}z'$ for some $z, z' \in Z$ holds iff $g \in \overline{P}$ (where $\overline{P} = N_G(\overline{S})$). The map $G *_{\overline{P}} \mathcal{N}^*X \to T^*X$ is generically finite.*

First proof. As it was shown $\widetilde{P} \supset \overline{P}$. Since $\widetilde{P}$ normalizes the family of orbits $\overline{U}z$ it also normalizes $\mathcal{N}^*X$ the conormal bundle to this foliation. So the composition map

$$G *_{\widetilde{P}} \mathcal{N}^*X \to G\mathcal{N}^*X \subset T^*X$$

is well defined and dominant. Calculating the dimensions:

$$\dim \mathcal{N}^*X = \dim Z + \dim \overline{U}z + \operatorname{codim}_X \overline{U}z = \dim T^*X - \dim P_u$$

$$\dim \widetilde{P} \geqslant \dim \overline{P} = \dim L + \dim P_u \cap M + \dim \overline{Q}_u = \dim P$$

$$\dim G *_{\widetilde{P}} \mathcal{N}X = \dim G/\widetilde{P} + \dim \mathcal{N}^*X \leqslant \dim T^*X$$

we get that the considered map can be dominant only when $\dim G *_{\widetilde{P}} \mathcal{N}^*X = \dim T^*X$ (where $z \in Z$). This can happen only when $\widetilde{P} = \overline{P}$ and in this case $G *_{\overline{P}} \mathcal{N}^*X \to T^*X$ is generically finite. $\qquad\square$

Second proof. Let $\widetilde{S}$ be the normalizer of $\overline{U}z$. Let us denote by $\widetilde{L}_0$ the Levi subgroup of $\widetilde{S}$ that contains L_0 and is normalized by T (by Lemma 5.21 T normalizes $\widetilde{S}$). Let $\widetilde{S}_z$ be the stabilizer in $\widetilde{S}$ of the point z, then we have $\widetilde{S} = \overline{U}\widetilde{S}_z$. Taking the quotient of this equality by the unipotent radical $\widetilde{S}_u$ of $\widetilde{S}$ (we note that $\widetilde{S}_u \subset \overline{U}$) we obtain that the image of the unipotent group $\overline{U}$ in $\widetilde{L}_0$ acts transitively on the homogeneous space $\widetilde{L}_0/\operatorname{Im}\widetilde{S}_z$ of the reductive group $\widetilde{L}_0$ (where we denoted by $\operatorname{Im}\widetilde{S}_z$ the image of $\widetilde{S}_z$ in $\widetilde{S}/\widetilde{S}_u$). From the fact that a variety homogeneous under a reductive group and a unipotent group is a point, we get that the homogeneous space $\widetilde{L}_0/\operatorname{Im}\widetilde{S}_z$ is a point and $\operatorname{Im}\widetilde{S}_z = \widetilde{L}_0$. From the Levi decomposition we get that $\widetilde{S}_z$ contains a subgroup conjugated to $\widetilde{L}_0$ by an element of $s_u \in \widetilde{S}_u \subset \overline{U}$. Thus taking the point $s_u z$ instead of z we can assume that $\widetilde{S}_z$ contains $\widetilde{L}_0$. Since the action of P_u is free on X° the intersection $\widetilde{L}_0 \cap P_u \subset \widetilde{S}_z \cap P_u$ must be trivial, which implies $\widetilde{L}_0 = L_0$. $\qquad\square$

6. Horospherical cotangent bundle

In this section we shall define a variety of degenerate horospheres $\mathcal{H}or$. Our aim is to prove that the conormal bundle to the family of degenerate horospheres maps birationally onto the cotangent bundle of $\mathcal{H}or$. This is a generalization of a theorem proved by E.B.Vinberg [18, §5 Thm.3].

Consider the G-translates of horospheres from the foliation constructed in Theorem 5.5. By Theorem 5.22 we can identify this set with the variety $\mathcal{H}or := G *_{\overline{P}} Z$. Since $\dim P = \dim \overline{P}$ we have $\dim \mathcal{H}or = \dim X$. Let us define the incidence variety:

$$\mathcal{U} := \{(x, \mathcal{H}) \in X \times \mathcal{H}or | x \in \mathcal{H}\}.$$

We note that a general point of X is contained in some $\mathcal{H} \in \mathcal{H}or$ (since GZ is dense in X). Thus the projection $p_X : \mathcal{U} \to X$ is dominant. The variety $\mathcal{U}$ can be identified with the subvariety $G *_{\overline{P}} \mathcal{U}_0$ of $G *_{\overline{P}} (X \times Z)$ (here $\overline{P}$ is acting diagonally on $X \times Z$ via a standard action on X and via the quotient $\overline{P}/S$ on Z) where

$$\mathcal{U}_0 = \{(x, z) \in X \times Z | x \in \overline{U}z\}.$$

Let us notice that Z can be diagonally embedded in $\mathcal{U}_0$.

For this incidence variety following Vinberg we can define the skew conormal bundle $SN^*\mathcal{U}$, that we denote by HT^*X (for details see [18, Section 4], [17, Section 2]).

The variety HT^*X can be identified with the variety of following triples $(x, \xi, \mathcal{H})$:

$$x \in \mathcal{H} \in \mathcal{H}or, \quad \xi \in T_x^*X, \quad \xi = 0|_{T_x\mathcal{H}}.$$

From Theorem 5.22 we also get that HT^*X is identified with $G *_{\overline{P}} \mathcal{N}$.

Consider the following commutative diagram:

$$
\begin{array}{ccccc}
T^*X & \xleftarrow{\widehat{p}_X} & HT^*X & \xrightarrow{\widehat{p}_{\mathcal{H}or}} & T^*\mathcal{H}or \\
\downarrow & & \downarrow & & \downarrow \\
X & \xleftarrow{p_X} & \mathcal{U} & \xrightarrow{p_{\mathcal{H}or}} & \mathcal{H}or
\end{array}
$$

Theorem 5.22 can be restated in the following form:

Theorem 6.1. *The morphism $HT^*X \xrightarrow{\widehat{p}_X} T^*X$ is generically finite.*

We are ready to prove the following generalization of a result of Vinberg [18, Thm.3].

Theorem 6.2. *The morphism $HT^*X \xrightarrow{\widehat{p}_{\mathcal{H}or}} T^*\mathcal{H}or$ is birational.*

Proof. Since the $T^*\mathcal{H}or$ is a vector bundle over $\mathcal{H}or$ and HT^*X maps dominantly onto $\mathcal{H}or$ it is sufficient to prove the claim of the theorem fiberwise. Let $\mathcal{H} \in \mathcal{H}or$, then the fiber of HT^*X over $\mathcal{H}$ is identified with $\mathcal{N}^*\mathcal{H}$ — the conormal bundle to $\mathcal{H} \subset X$. We shall prove that image of $\mathcal{N}^*\mathcal{H}$ under $\widehat{p}_{\mathcal{H}or}$ is birationally isomorphic to $T_{\mathcal{H}}^*\mathcal{H}or$. Since all the maps are G-equivariant we can assume that $\mathcal{H} \in Z$, i.e. $\mathcal{H} = (P_u \cap \overline{Q})x$ for some $x \in Z$. We notice that the $P_u \cap \overline{Q}$-action on $\mathcal{N}^*\mathcal{H}$ is free and the fiber $\mathcal{N}_x^*X$ of the conormal bundle to $\mathcal{H}$ at some point $x \in \mathcal{H}$ defines an obvious section of this action. Without loss of generality we can shrink Z to get an open subset isomorphic to $A \times C$, so we can chose $x \in C$. By [18, Section 4] the morphism $\widehat{p}_{\mathcal{H}or}$ maps the fiber $\mathcal{N}_x^*X$ isomorphically to the subspace $N_{\mathcal{H}}^*\mathcal{H}or_x \subset T_{\mathcal{H}}^*\mathcal{H}or$, where $\mathcal{H}or_x = p_X^{-1}(x)$ is the set of horospheres containing x, and $N_{\mathcal{H}}^*\mathcal{H}or_x$ is the fiber over $\mathcal{H}$ of the conormal bundle to $\mathcal{H}or_x$ in $\mathcal{H}or$. (Let us also notice that since $p_{\mathcal{H}or}$ and p_X are both surjective and the dimension of the general fiber of $p_{\mathcal{H}or}$ is $\dim(\mathfrak{p}_u \cap \overline{\mathfrak{q}})$ we have $\dim \mathcal{H}or_x = \dim(\mathfrak{p}_u \cap \overline{\mathfrak{q}})$.) From the above it is clear that birationality of $\widehat{p}_{\mathcal{H}or}$ will follow from the following proposition.

Proposition 6.3. *The $P_u \cap \overline{Q}$-action is generically free on $T^*_{\mathcal{H}}\mathcal{H}or$. The subspace $N^*_{\mathcal{H}}\mathcal{H}or_x$ intersects a general $P_u \cap \overline{Q}$-orbit from $T^*_{\mathcal{H}}\mathcal{H}or$ transversally in a single point.*

Remark 6.4. The action of $P_u \cap \overline{Q}$ on $T^*_{\mathcal{H}}\mathcal{H}or$ is well defined since $P_u \cap \overline{Q}$ stabilizes $\mathcal{H}$.

Proof. We note that the tangent space $T_{\mathcal{H}}\mathcal{H}or$ can be identified with $\mathfrak{g}/\overline{\mathfrak{s}} \times T_x C$, and the cotangent space $T^*_{\mathcal{H}}\mathcal{H}or$ is isomorphic to

$$\overline{\mathfrak{s}}^{\perp} \times T^*_x C \cong (\mathfrak{a} + \mathfrak{p}_u \cap \overline{\mathfrak{m}} + \overline{\mathfrak{q}}_u) \times T^*_x C.$$

From this description we see that our problem is reduced to the study of $P_u \cap \overline{Q}$-orbits in $\overline{\mathfrak{s}}^{\perp}$. Now Lemma 5.16 finishes the proof of the first part of the proposition.

Let us consider a subvariety $\mathcal{H}or_{tr}$ of the variety of horospheres $\mathcal{H}or$ that is equal to $(P_u \cap \overline{Q}_u^-) \times Z$. It defines the family of horospheres $\mathcal{U}|_{\mathcal{H}or_{tr}} = p^{-1}_{\mathcal{H}or}(\mathcal{H}or_{tr})$ that maps isomorphically under p_X to X_0. Indeed it consists of translates by the elements of $P_u \cap \overline{Q}_u^-$ of the orbits $(P_u \cap \overline{Q})z$ for $z \in Z$, and these translates do not intersect pairwise (as follows from the freeness of the P_u-action and the equality $(P_u \cap \overline{Q}_u^-)(P_u \cap \overline{Q}) = P_u$ that is a corollary of [7, Prop.28.7]). Hence this family maps isomorphically to the open set $(P_u \cap \overline{Q}_u^-)(P_u \cap \overline{Q})Z = P_u Z = X^{\circ}$. If we consider the embedding of Z in $\mathcal{U}$, we get $p^{-1}_{\mathcal{H}or}(\mathcal{H}or_{tr}) = P_u Z$.

Claim We can assume that for sufficiently general $x \in Z$ the subvarieties $\mathcal{H}or_{tr}$ and $\mathcal{H}or_x$ of the variety $\mathcal{H}or$ are transversal in the point $\mathcal{H}$ corresponding to the horosphere $\overline{U}x$. We also have the equality for tangent spaces in the point $\mathcal{H} \in \mathcal{H}or$:

$$T_{\mathcal{H}}\mathcal{H}or = T_{\mathcal{H}}\mathcal{H}or_x \oplus T_{\mathcal{H}}\mathcal{H}or_{tr} \qquad (*).$$

Proof. Shrinking the varieties $X, \mathcal{H}or, \mathcal{U}$ we can assume that they are smooth G-varieties and the morphisms $p_X, p_{\mathcal{H}or}$ are submersive. Let us notice that the horospheres parameterized by $\mathcal{H}or_{tr}$ do no intersect each other and cover the open subset X°. This implies that $p^{-1}_{\mathcal{H}or}(\mathcal{H}or_{tr})$ maps isomorphically to X° under p_X. Since p_X is submersive, $p^{-1}_{\mathcal{H}or}(\mathcal{H}or_{tr})$ also intersects each fiber $\mathcal{H}or_x = p^{-1}_X(x)$ for each $x \in X^{\circ}$ transversally, exactly in one point. Since each fiber $\mathcal{H}or_x$ maps immersively into $\mathcal{H}or$ under $p_{\mathcal{H}or}$, this implies the transversality of $\mathcal{H}or_{tr}$ and $\mathcal{H}or_x$ in $\mathcal{H}or$. The varieties $p^{-1}_{\mathcal{H}or}(\mathcal{H}or_{tr})$ and $\mathcal{H}or_x$ have complementary dimensions in $\mathcal{U}$ so the varieties $\mathcal{H}or_{tr}$ and $\mathcal{H}or_x$ have complementary dimension in $\mathcal{H}or$ that implies $(*)$. $\square$

Lemma 6.5. *$T_{\mathcal{H}}\mathcal{H}or_{tr}$ is identified canonically with $\mathfrak{p}_u \cap \overline{\mathfrak{q}}_u^- \oplus T_x Z \subset \mathfrak{g}/\overline{\mathfrak{s}} \times T_x C$ and the fiber of the conormal bundle $N^*_{\mathcal{H}}\mathcal{H}or_{tr}$ is identified with $\mathfrak{p}_u \cap \overline{\mathfrak{q}} \subset \overline{\mathfrak{s}}^{\perp} \times T^*_x C$.*

Proof. The first assertion is trivial. Since $T_{\mathcal{H}}\mathcal{H}or_{tr} \cong \mathfrak{p}_u \cap \overline{\mathfrak{q}}_u^- \oplus T_{\mathcal{H}}Z \subset \mathfrak{g}/\overline{\mathfrak{s}} \times T_x C$, the linear space $N^*_{\mathcal{H}}\mathcal{H}or_{tr}$ is identified with the subspace $\mathfrak{p}_u \cap \overline{\mathfrak{q}} \subset \overline{\mathfrak{s}}^{\perp}$ which consists of the linear functions on $\mathfrak{g}/\overline{\mathfrak{s}}$ annihilated on $\mathfrak{p}_u \cap \overline{\mathfrak{q}}_u^-$. $\square$

Dualizing the equality $(*)$ we get:

$$T^*_{\mathcal{H}}\mathcal{H}or = N^*_{\mathcal{H}}\mathcal{H}or_x \oplus N^*_{\mathcal{H}}\mathcal{H}or_{tr} = N^*_{\mathcal{H}}\mathcal{H}or_x \oplus (\mathfrak{p}_u \cap \overline{\mathfrak{q}}) \qquad (**)$$

We shall be finished after proving the next proposition:

Proposition 6.6. *The intersection of $N^*_{\mathcal{H}}\mathcal{H}or_x$ and a general $P_u \cap \overline{Q}$-orbit from $\overline{\mathfrak{s}}^{\perp} \oplus T^*_x C$ consists of a single point.*

Proof. Let us take a sufficiently general point $\xi \in N^*_{\mathcal{H}}\mathcal{H}or_x$ (we assume that the projection to $\mathfrak{a}$ is sufficiently general). We shall prove that $\mathrm{Ad}(u)\xi \notin N^*_{\mathcal{H}}\mathcal{H}or_x$ for nontrivial $u \in P_u \cap \overline{Q}$. We represent u via exponential map $u = \exp(\eta)$,

where $\eta = \sum_{\alpha \in \Delta(\mathfrak{p}_u \cap \overline{\mathfrak{q}})} c_\alpha e_\alpha \in \mathfrak{p}_u \cap \overline{\mathfrak{q}}$. Consider the one-parameter subgroup $\lambda :$ $\mathbb{K}^\times \to Z(L_0)$ from the proof of Theorem 5.5. We recall that λ is nonnegative on $\overline{\mathfrak{s}}^\perp = \mathfrak{a} + \mathfrak{p}_u \cap \overline{\mathfrak{m}} + \overline{\mathfrak{q}}_u$ and $\mathfrak{a} + \mathfrak{p}_u \cap \overline{\mathfrak{m}}$ is the component of $\overline{\mathfrak{s}}^\perp$ of zero λ-weight. Since λ lies in the stabilizer of $x \in Z$ it preserves the subvariety $\mathcal{H}or_x$; it also stabilizes the horosphere $\mathcal{H} = (P_u \cap \overline{Q})x$. Consequently, λ acts on the linear space $N^*_{\mathcal{H}}\mathcal{H}or_x$. Let us choose a strictly dominant one-parameter subgroup $\lambda_0 : \mathbb{K}^\times \to T$ (in particular $\langle \lambda_0, \alpha \rangle > 0$ for all $\alpha \in \Delta(\mathfrak{p}_u)$).

Consider the set of $\alpha \in \Delta(\mathfrak{p}_u \cap \overline{\mathfrak{q}})$ such that $c_\alpha \neq 0$ and the value $\ell = \langle \lambda, \alpha \rangle$ is the least possible. We choose some γ from this set with the smallest value $\langle \lambda_0, \gamma \rangle$. For the vectors ξ, η by ξ_ℓ, η_ℓ we denote the components of weight ℓ. Let us prove the following lemma.

Lemma 6.7. *For the vector* $\mathrm{Ad}(u)\xi - \xi$ *the component of weight* ℓ *with respect to* λ *is nonzero but its* T*-equivariant projection to the subspace* $\mathfrak{a} + \mathfrak{p}^-_u \cap \overline{\mathfrak{q}}_u$ *is zero.*

Proof. First assume that $\ell \neq 0$. Denote by $\xi_\mathfrak{a}$ and by $\xi_{\mathfrak{p}_u \cap \overline{\mathfrak{m}}}$ the projection of ξ to $\mathfrak{a}$ and $\mathfrak{p}_u \cap \overline{\mathfrak{m}}$ respectively. The component of ξ of λ-weight zero is equal to $\xi_0 = \xi_\mathfrak{a} + \xi_{\mathfrak{p}_u \cap \overline{\mathfrak{m}}}$.

Let us notice that

$$\mathrm{Ad}(u)\xi = \xi + [\eta, \xi] + \frac{(\sum_{\alpha \in \Delta(\mathfrak{p}_u \cap \overline{\mathfrak{q}})} c_\alpha \mathrm{ad}(e_\alpha))^2}{2!}\xi + \dots$$

The weights in the λ-weight decomposition of ξ are nonnegative. Since ℓ has minimal possible value on the root subspaces in the exponential decomposition of u and application of each e_α increases the λ-weight by $\langle \lambda, \alpha \rangle \geqslant \ell$, then the component of $\mathrm{Ad}(u)\xi$ of weight ℓ (which we denote by $(\mathrm{Ad}(u)\xi)_\ell$), does not contain the summands that consist of the monomials on $\mathrm{ad}(e_\alpha)$ applied to the components of ξ with the λ-weight > 0 or the monomials of degree > 1 applied to ξ_0. This implies that $(\mathrm{Ad}(u)\xi)_\ell = \xi_\ell + [\eta_\ell, \xi_0]$. Since $\eta_\ell \in \mathfrak{p}_u$ and $\xi_0 \in \mathfrak{a} + \mathfrak{p}_u$ then $[\eta_\ell, \xi_0] \in \mathfrak{p}_u$. In particular $[\eta_\ell, \xi_0]$ has zero projection to $\mathfrak{a} + \mathfrak{p}^-_u \cap \overline{\mathfrak{q}}_u$; this proves the second part of the lemma in the case $\ell \neq 0$. From the above we also see that the component of $\mathrm{Ad}(u)\xi - \xi$ of weight γ is equal to $[e_\gamma, \xi_\mathfrak{a}]$. Indeed, there are no other components of this T-weight since the application of elements $\mathrm{ad}(e_\alpha)$ (with the α such that $\langle \lambda_0, \alpha \rangle \geqslant \langle \lambda_0, \gamma \rangle$) to the components of ξ which belong to $\mathfrak{p}_u \cap \overline{\mathfrak{m}}$ (that have λ_0-weight strictly bigger than zero) give rise to a component with λ_0-weight strictly bigger than $\langle \lambda_0, \gamma \rangle$. The component $[e_\gamma, \xi_\mathfrak{a}]$ is nonzero since $(\mathfrak{p}_u \cap \overline{\mathfrak{q}}) \cap \mathfrak{z}_\mathfrak{g}(\mathfrak{a}) = 0$.

If $\ell = 0$ we see that the projection of $(P_u \cap \overline{Q})\xi$ to the component of zero λ-weight is equal to $(P_u \cap \overline{M})\xi_0 = \xi_0 + \mathfrak{p}_u \cap \overline{\mathfrak{m}}$ and the stabilizer of ξ_0 in $P_u \cap \overline{M}$ is trivial. This implies that $\mathrm{Ad}(u)\xi - \xi$ has nonzero projection to $\mathfrak{p}_u \cap \overline{\mathfrak{m}}$ and its projection to $\mathfrak{a}$ is trivial. $\qquad\square$

Assume that $\xi, \mathrm{Ad}(u)\xi \in N^*_{\mathcal{H}}\mathcal{H}or_x$. Since $N^*_{\mathcal{H}}\mathcal{H}or_x$ is λ-invariant the components ξ_ℓ and $(\mathrm{Ad}(u)\xi)_\ell$ of weight ℓ with respect to λ also belong to $N^*_{\mathcal{H}}\mathcal{H}or_x$. By the previous Lemma ξ_ℓ and $(\mathrm{Ad}(u)\xi)_\ell$ are different vectors with the same projections to the subspace $\mathfrak{a} + \mathfrak{p}^-_u \cap \overline{\mathfrak{q}}_u$. We note that ξ and $u\xi$ have the same projection to $T^*_x C$ that implies that ξ_ℓ and $(\mathrm{Ad}(u)\xi)_\ell$ have the same projection to the subspace $(\mathfrak{a} + \mathfrak{p}^-_u \cap \overline{\mathfrak{q}}_u) \oplus T^*_x C$. By $(**)$ the linear subspace $N^*_{\mathcal{H}}\mathcal{H}or_x$ projects isomorphically to $(\mathfrak{a} + \mathfrak{p}^-_u \cap \overline{\mathfrak{q}}_u) \oplus T^*_x C$. In particular there are no distinct vectors in $N^*_{\mathcal{H}}\mathcal{H}or_x$ with the same projection to $(\mathfrak{a} + \mathfrak{p}^-_u \cap \overline{\mathfrak{q}}_u) \oplus T^*_x C$. We come to contradiction that proves the proposition. $\qquad\square$

We have the following corollary of the proof of Proposition 6.6.

Proposition 6.8. *Let* $\mathcal{N}^*_z X$ *be the fiber of the conormal bundle to the foliation of degenerate horospheres at some point* $z \in Z$ *and* $V_z \subseteq \mathcal{N}^*_z X$ *be a* λ*-invariant subspace*

such that $\mu_X(V_z)$ is mapped isomorphically to $\mathfrak{a} + \mathfrak{p}_u^- \cap \overline{\mathfrak{q}}_u$ under the T-equivariant projection to this subspace. Then the intersection of $\mu_X(V_z)$ and a general $P_u \cap \overline{Q}$-orbit from $\overline{\mathfrak{s}}^\perp$ consists of a single point.

Proof. The proof of Proposition 6.6 goes word by word if we use that $\mu_X(V_z)$ is λ-invariant and maps isomorphically to $\mathfrak{a} + \mathfrak{p}_u^- \cap \overline{\mathfrak{q}}_u$ under the T-equivariant projection. $\qquad\square$

$\square$

$\square$

7. The Little Weyl group.

By Theorem 5.22 we have the generically finite map $G *_{\overline{P}} \mathcal{N}^* X \to T^* X$ which in fact is a rational Galois covering. The aim of this section is to prove that the Galois group of this covering is equal to the little Weyl group of X introduced by Knop in [9].

Consider the following commutative diagram.

$$
\begin{array}{ccccc}
G *_{\overline{P}} \mathcal{N}^* X & \xrightarrow{\mu_{\mathcal{N}^*}} & G *_{\overline{P}} (\mathfrak{a} + \overline{\mathfrak{p}}_u) & \longrightarrow & \mathfrak{a} \\
\downarrow & & \downarrow & & \downarrow \\
T^* X & \xrightarrow{\mu_X} & G(\mathfrak{a} + \overline{\mathfrak{p}}_u) & \longrightarrow & \mathfrak{t}/W
\end{array}
$$

Here the upper right horizontal arrow is a rational quotient by the group G. The lower right arrow is the composition of the categorical quotient by G and the Chevalley isomorphism $\mathfrak{g}/\!\!/G = \mathfrak{t}/W$. The central vertical arrow is $[g * \xi] \to g\xi$. The map $\mu_{\mathcal{N}^*}$ is the following $[g * \xi] \to [g * \mu_X(\xi)]$. It is well defined since μ_X is G-equivariant and in particular $\overline{P}$-equivariant.

The little Weyl group can be defined as follows, after F.Knop [10, 9]. Consider the fiber product $\mathfrak{a} \times_{\mathfrak{t}/W} T^* X$. In general it is not irreducible. There is a natural embedding of $\mathcal{N}^* X$ in $\mathfrak{a} \times_{\mathfrak{t}/W} T^* X$, for $\eta \in \mathcal{N}^* X$ it is defined by taking the semisimple part of the Jordan decomposition for $\mu_X(\eta)$ in $\mathfrak{a}$ and by embedding η in $T^* X$. Let us denote by $\widehat{T^* X}$ an irreducible component of $\mathfrak{a} \times_{\mathfrak{t}/W} T^* X$ containing the image of $\mathcal{N}^* X$. Define the action of the Weyl group $N_G(\mathfrak{a})/Z_G(\mathfrak{a})$ on $\mathfrak{a} \times_{\mathfrak{t}/W} T^* X$ by its action on the left multiple.

Definition 7.1. The maximal subgroup W_X of $N_G(\mathfrak{a})/Z_G(\mathfrak{a})$ that preserves the irreducible component $\widehat{T^* X}$ is called the little Weil group of X.

The aim of this section is to prove the following:

Theorem 7.2. *The map $G *_{\overline{P}} \mathcal{N}^* X \to T^* X$ is a rational Galois covering with group W_X.*

To prove the Theorem 7.2 we need the notion of normalized moment map $\widetilde{\mu}_X : T^* X \to M_X$ introduced by Knop in [9]. It can be defined via taking the Stein factorization $T^* X \to M_X \to G(\mathfrak{a} + \mathfrak{p}_u^-)$ of the moment map μ_X. In other words we take for M_X a normalization of $G(\mathfrak{a} + \mathfrak{p}_u^-)$ in the field of rational functions of $T^* X$. We remind the reader that for the horospherical variety G/P_0^- the variety $G *_{P^-} (\mathfrak{a} + \mathfrak{p}_u^-)$ is a G-birational model for M_{G/P_0^-} (see [9, §4]).

The next lemma provides different birational G-models of M_{G/P_0^-}.

Lemma 7.3. *The varieties $G *_{P^-} (\mathfrak{a} + \mathfrak{p}_u^-)$ and $G *_{\overline{P}} (\mathfrak{a} + \overline{\mathfrak{p}}_u)$ are birationally isomorphic to $G *_M (\mathfrak{a} + M *_{M \cap P^-} (\mathfrak{p}_u^- \cap \mathfrak{m}))$ as G-varieties.*

Proof. By Proposition 5.4 there exists $\xi_n \in \mathfrak{p}_u^- \cap \mathfrak{m}$ such that $(P^- \cap M)\xi_n$ is dense in $\mathfrak{p}_u^- \cap \mathfrak{m}$. Since $P_u^-(\mathfrak{a} + (\mathfrak{p}_u^- \cap \mathfrak{m}))$ is dense in $\mathfrak{a} + \mathfrak{p}_u^-$ we get that $P^-(\mathfrak{a} + \xi_n) \supset P_u^-((P^- \cap M)(\mathfrak{a} + \xi_n))$ is also dense there. This implies that $\mathfrak{a} + \xi_n$ intersects a general P^--orbit from $\mathfrak{a} + \mathfrak{p}_u^-$ in at most one point. Since $Z_{P^-}(\mathfrak{a} + \xi_n) = Z_{M \cap P^-}(\xi_n)$ we get that the map $P^- *_{Z_{M \cap P^-}(\xi_n)} (\mathfrak{a} + \xi_n) \to \mathfrak{a} + \mathfrak{p}_u$ is birational thus $G *_{P^-} (\mathfrak{a} + \mathfrak{p}_u^-)$ is G-birational to $G *_{P^-} (P^- *_{Z_{M \cap P^-}(\xi_n)} (\mathfrak{a} + \xi_n)) \cong G *_{Z_{M \cap P^-}(\xi_n)} (\mathfrak{a} + \xi_n)$ that is contained in $G *_M (\mathfrak{a} + M *_{M \cap P^-} (\mathfrak{p}_u^- \cap \mathfrak{m}))$ as the dense subset.

Using the equality $(\overline{P} \cap M) = (P^- \cap M)$ and repeating literally all the above arguments for the group $\overline{P}$ instead of P^-, we see that $\mathfrak{a} + \xi_n$ also provides a section for the action of $\overline{P}$ on $(\mathfrak{a} + \overline{\mathfrak{p}}_u)$ and we get G-birational isomorphism between $G *_{\overline{P}} (\mathfrak{a} + \overline{\mathfrak{p}}_u)$ and $G *_M (\mathfrak{a} + M *_{M \cap \overline{P}} (\overline{\mathfrak{p}}_u \cap \mathfrak{m}))$. $\qquad\square$

First let us fit the conormal bundle $\mathcal{N}^* X$ into the family $\mathcal{N}^* \mathcal{X}$ of conormal bundles in the fibers of $\mathcal{X} \to \mathbb{A}^1$. We have the following proposition.

Proposition 7.4. *The general fibers of the morphisms $\mu_X : \mathcal{N}^* X \to \mathfrak{a} + \overline{\mathfrak{p}}_u$, $\mu_{\mathcal{X}} : \mathcal{N}^* \mathcal{X} \to \mathfrak{a} + \overline{\mathfrak{p}}_u$, $\mu_{\mathcal{N}^*} : G *_{\overline{P}} \mathcal{N}^* X \to G *_{\overline{P}} (\mathfrak{a} + \overline{\mathfrak{p}}_u)$ and $\mu_{\mathcal{N}^* \mathcal{X}} : G *_{\overline{P}} \mathcal{N}^* \mathcal{X} \to G *_{\overline{P}} (\mathfrak{a} + \overline{\mathfrak{p}}_u)$ are irreducible.*

Proof. We shall need the following lemma.

Lemma 7.5. *Let X be a normal variety and $f : X \to Y$ be a dominant morphism. Assume we have a rational section of f, i.e. $\sigma : Y \dashrightarrow X$, such that $f \circ \sigma = id_Y$. Then the general fiber of f is irreducible.*

Proof. Consider a variety $\widetilde{Y}$ that is equal to the normalization of Y in the field of rational functions on X. Then we have two morphisms $\widetilde{f} : X \to \widetilde{Y}$, and $\pi : \widetilde{Y} \to Y$ such that $f = \pi \circ \widetilde{f}$, the general fiber of $\widetilde{f}$ is irreducible, and π is finite. Then the composition $\widetilde{f} \circ \sigma$ gives a birational section of the finite morphism π, that proves that π is birational and gives the irreducibility of the general fiber of f. $\qquad\square$

To prove an irreducibility of the general fiber for $\mu_X|_{\mathcal{N}^* X}$ by Lemma 7.5 it is sufficient to construct a rational section $\mathfrak{a} + \overline{\mathfrak{p}}_u \dashrightarrow \mathcal{N}^* X$ of the morphism $\mu_X|_{\mathcal{N}^* X} : \mathcal{N}^* X \to \mathfrak{a} + \overline{\mathfrak{p}}_u$. Let us notice that by Proposition 5.13 and Remark 5.14 there exists a subspace $V_z \in \mathcal{N}_z^* X$ such that $\mu_X(V_z)$ is isomorphic to V_z and $\mu_X(V_z)$ projects isomorphically to $\mathfrak{a} + \mathfrak{p}_u^- \cap \overline{\mathfrak{q}}_u$ under the T-equivariant projection from $\overline{\mathfrak{s}}^\perp$ to $\mathfrak{a} + \mathfrak{p}_u^- \cap \overline{\mathfrak{q}}_u$ with the fibers parallel to the subspace $\mathfrak{p}_u \cap \overline{\mathfrak{q}}$. By Proposition 6.8 the map $(P_u \cap \overline{Q}) \times \mu_X(V_z) \to \mathfrak{a} + \overline{\mathfrak{p}}_u$ is birational. Since the map $(P_u \cap \overline{Q}) \times V_z \to (P_u \cap \overline{Q}) V_z \subset \mathcal{N}^* X$ is an isomorphism onto its image, from the G-equivariance of μ_X the variety $(P_u \cap \overline{Q}) V_z$ defines a rational section of $\mu_X : \mathcal{N}^* X \to \mathfrak{a} + \overline{\mathfrak{p}}_u$. The general fibers for $\mu_{\mathcal{X}}|_{\mathcal{N}^* \mathcal{X}}$ are irreducible since the section constructed above is a section for this morphism as well.

The irreducibility of general fibers for $\mu_{\mathcal{N}^*}$ and $\mu_{\mathcal{N}^* \mathcal{X}}$ now follows from the next trivial lemma:

Lemma 7.6. *Let $G \supset H$ be algebraic groups, X, Y be normal H-varieties and $f_H : X \to Y$ be an H-morphism. Let $f_G : G *_H X \to G *_H Y$ be a morphism defined by $[g * x] \mapsto [g * f_H(x)]$ then $f_G^{-1}(f_G([g * x])) = g * f_H^{-1}(f_H(x))$.*

$\qquad\square$

Let us denote by Θ the irreducible component of the intersection of $\mu_X^{-1}(\mathfrak{a}^{pr} + (\mathfrak{p}_u^- \cap \mathfrak{m}))$ and $\mathcal{N}^* X$ that maps dominantly to $\mathfrak{a} + (\mathfrak{p}_u^- \cap \mathfrak{m})$ (it is unique by Proposition 7.4). Consider $\Sigma = M\Theta$, it is a component of $\mu_X^{-1}(\mathfrak{a}^{pr} + M(\mathfrak{p}_u^- \cap \mathfrak{m}))$ that maps dominantly to $\mathfrak{a} + M(\mathfrak{p}_u^- \cap \mathfrak{m})$ and intersects $\mathcal{N}^* X$. Since $G(\mathfrak{a}^{pr} + M(\mathfrak{p}_u^- \cap \mathfrak{m}))$ is dense in $\mu_X(T^* X)$ we get that $G\Sigma$ is dense in $T^* X$. Let $\xi \in \mathfrak{a}^{pr} + M(\mathfrak{p}_u^- \cap \mathfrak{m})$, if

$\mathrm{Ad}(g)\xi \in (\mathfrak{a}^{pr} + M(\mathfrak{p}_u^- \cap \mathfrak{m}))$ for some $g \in G$ from the uniqueness of the Jordan decomposition we get that the semisimple parts of ξ and $\mathrm{Ad}(g)\xi$ are conjugated by g and both lie in $\mathfrak{a}^{pr}$. Thus the set $\{g \in G \mid g\Sigma \cap \Sigma \neq \emptyset\}$ is contained in a finite union of the cosets of M in $N_G(\mathfrak{a})$. By [18, Lemma 2] the morphism $G *_M \Sigma \to T^*X$ is a rational Galois covering with the Galois group N_X/M, where

$$N_X := \{g \in N_G(\mathfrak{a}) \mid g\overline{\Sigma} = \overline{\Sigma}\},$$

and the action of N_X/M is defined by $nM \circ [g * z] = [gn^{-1} * nz]$ for $n \in N_X$.

Theorem 7.7. *The varieties T^*X and M_X are G-birationally isomorphic to $G *_{N_X} \Sigma$ and $G *_{N_X} (\mathfrak{a}^{pr} + M *_{P^- \cap M} (\mathfrak{p}_u^- \cap \mathfrak{m}))$. The map $\Phi : M *_{P^- \cap M} \Theta \to M\Theta$ is a birational isomorphism and the normalized moment map $\widetilde{\mu}_X : T^*X \to M_X$ is described on some open subset by the formula $\widetilde{\mu}_X([g * \eta]) = [g * \mu_{\mathcal{N}^*}(\Phi^{-1}(\eta))]$.*

Proof of the Theorem 7.7. To prove that the morphism Φ is birational it is sufficient to show that it is isomorphism on the open subset $(P_u \cap M) \times \Theta$ in $M *_{P^- \cap M} \Theta$. Let us notice that the projection of $\mathcal{N}^*X$ to X is equal to $(P_u \cap \overline{Q})Z$ and $\overline{Q} \cap (P_u \cap M) = \{e\}$. Thus we have an isomorphism $(P_u \cap M)(P_u \cap \overline{Q})Z \cong (P_u \cap M) \times (P_u \cap \overline{Q})Z$ which implies $(P_u \cap M)\mathcal{N}^*X \cong (P_u \cap M) \times \mathcal{N}^*X$. In particular $(P_u \cap M)\Theta \cong (P_u \cap M) \times \Theta$.

We have an M-equivariant rational map $\mu_{\mathcal{N}^*} \circ \Phi^{-1} : \mu_X^{-1}(\mathfrak{a}^{pr} + M(\mathfrak{p}_u^- \cap \mathfrak{m})) \dashrightarrow M *_{P^- \cap M} (\mathfrak{a}^{pr} + (\mathfrak{p}_u^- \cap \mathfrak{m}))$, that induces the rational map:

$$\widetilde{\mu}_X : G *_{N_X} \Sigma \dashrightarrow G *_{N_X} (\mathfrak{a}^{pr} + M *_{P^- \cap M} (\mathfrak{p}_u^- \cap \mathfrak{m})),$$

which factors the moment map μ_X. To prove that M_X is birational to $G *_{N_X} (\mathfrak{a}^{pr} + M *_{P^- \cap M} (\mathfrak{p}_u^- \cap \mathfrak{m}))$ it remains to prove irreducibility of general fibers for $\widetilde{\mu}_X$.

The irreducibility of general fibers for $\mu_X|_{\mathcal{N}^*X} \to \mathfrak{a} + \overline{\mathfrak{p}}_u$ (see Proposition 7.5) implies the irreducibility of general fibers for the maps $\mu_X|_\Theta, M *_{P^- \cap M} \Theta \to M *_{P^- \cap M} (\mathfrak{a}^{pr} + (\mathfrak{p}_u^- \cap \mathfrak{m}))$ and $\widetilde{\mu}_X$. $\qquad\square$

Remark 7.8. Since $G\mathcal{N}^*X$ is dense in T^*X to study the stabilizer of a general cotangent vector $\eta_x \in T_x^*X$ it suffice to consider $\eta_x \in \mathcal{N}^*X$, and acting by $\overline{P}$ we may assume that $\xi = \mu_X(\eta_x) \in \mathfrak{a}^{pr} + (\overline{\mathfrak{p}}_u \cap \mathfrak{m})$. From the description of M_X we get that $G_{\eta_x} \subset \overline{S} \cap G_\xi$.

Proof of the Theorem 7.2. As we have seen the Galois group of the rational covering $G *_{\overline{P}} \mathcal{N}^*X \to T^*X$ is equal to N_X/M and by Theorem 7.7 the map $M_{G/P_0^-} \to M_X$ is the quotient by the action of N_X/M which commutes with the G-action. By definition of Knop the little Weil group is the Galois group of the covering $M_{G/P_0^-} /\!\!/ G \to M_X /\!\!/ G$ (we note that $M_{G/P_0^-} /\!\!/ G \cong \mathfrak{a}$) which is also equal to N_X/M. $\qquad\square$

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